Since the zeros are \( x = -2, x = -1, x = 0, x = 1, \) and \( x = 2 \), the factors are \( x + 2, x + 1, x, x - 1, \) and \( x - 2 \). Thus \( P(x) = c(x + 2) (x + 1) x (x - 1) (x - 2) \). If we let \( c = 1 \), then \( P(x) = x^5 - 5x^3 + 4x \).

Since the zeros of the polynomial are \( 1, -2, \) and \( 3 \), it follows that \( P(x) = C(x + 1) (x - 1) (x - 2) = C(x^3 - 3x^2 + 4) \). Since \( P(0) = 4 \) we have \( 4 = 4C \) \( \iff \) \( C = 1 \). Thus \( P(x) = (x + 1) (x - 1) (x - 2) = x^3 - 3x^2 + 2x + 4 \).

The \( y \)-intercept is \( 2 \) and the zeros of the polynomial are \( -1, 1, \) and \( 2 \). It follows that \( P(x) = C(x + 2)^2 (x - 1)^2 = C(x^4 + 2x^3 - 3x^2 - 4x + 4) \). Since \( P(0) = 4 \) we have \( 4 = 4C \) \( \iff \) \( C = 1 \) and \( P(x) = x^4 + 2x^3 - 3x^2 - 4x + 4 \).

The \( y \)-intercept is \( 2 \) and the zeros of the polynomial are \( -2, -1, \) and \( 1 \). It follows that \( P(x) = C(x + 2) (x + 1)^2 (x - 1)^2 = C(x^4 + x^3 - 3x^2 - x + 2) \). Since \( P(0) = 2 \) we have \( 2 = 2C \) so \( C = 1 \) and \( P(x) = x^4 + x^3 - 3x^2 - x + 2 \).

By the Remainder Theorem, the remainder when \( P(x) = 6x^{1000} - 17x^{502} + 12x + 26 \) is divided by \( x + 1 \) is \( P(-1) = 6(-1)^{1000} - 17(-1)^{502} + 12(-1) + 26 = 6 - 17 + 12 + 26 = 3 \).

If \( x = -1 \) is a factor of \( Q(x) = x^{507} - 3x^{400} + 9^9 + 2 \), then \( Q(1) \) must equal \( 0 \). And \( Q(1) = (1)^{507} - 3(1)^{400} + 9^9 + 2 = 1 - 3 + 9 + 2 = 7 \neq 0 \), so \( x = -1 \) is not a factor.

So to calculate \( R(3) \), we start with \( 3 \), then subtract \( 2 \), multiply by \( 3 \), add \( 3 \), multiply by \( 3 \), subtract \( 2 \), multiply by \( 3 \), add \( 3 \), multiply by \( 3 \), and add \( 4 \), to get \( 157 \).

## 3.3 Real Zeros of Polynomials

1. \( P(x) = x^3 - 4x^2 + 3 \) has possible rational zeros \( \pm 1 \) and \( \pm 3 \).
2. \( Q(x) = x^4 - 3x^3 - 6x + 8 \) has possible rational zeros \( \pm 1, \pm 2, \pm 4, \pm 8 \).
3. \( R(x) = 2x^5 + 3x^3 + 4x^2 - 8 \) has possible rational zeros \( \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2} \).
4. \( S(x) = 6x^4 - x^2 + 2x + 12 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{3}{3}, \pm \frac{1}{4}, \pm \frac{3}{4} \).
5. \( T(x) = 4x^5 - 2x^2 - 7 \) has possible rational zeros \( \pm 1, \pm 7, \pm \frac{1}{2}, \pm \frac{7}{2}, \pm \frac{1}{4}, \pm \frac{7}{4} \).
6. \( U(x) = 12x^5 + 6x^3 - 2x - 8 \) has possible rational zeros \( \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{1}{6}, \pm \frac{1}{12} \).
7. (a) \( P(x) = 5x^3 - x^2 - 5x + 1 \) has possible rational zeros \( \pm 1, \pm \frac{1}{5} \).

(b) From the graph, the actual zeroes are \(-1, \frac{1}{5}, 1\).

8. (a) \( P(x) = 3x^3 + 4x^2 - x - 2 \) has possible rational zeros \( \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3} \).

(b) From the graph, the actual zeroes are \(-1 \text{ and } \frac{2}{3}\).

9. (a) \( P(x) = 2x^4 - 9x^3 + 9x^2 + x - 3 \) has possible rational zeros \( \pm 1, \pm 3, \pm \frac{1}{3}, \pm \frac{3}{2} \).

(b) From the graph, the actual zeroes are \(-\frac{1}{3}, 1, \text{ and } 3\).

10. (a) \( P(x) = 4x^4 - x^3 - 4x + 1 \) has possible rational zeros \( \pm 1, \pm \frac{1}{4}, \pm \frac{1}{2} \).

(b) From the graph, the actual zeroes are \(-\frac{1}{2}, 1, \text{ and } 3\).

11. \( P(x) = x^3 + 3x^2 - 4 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4 \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -x^3 + 3x^2 - 4 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
  & 1 & 3 & 0 & -4 \\
---&---&---&---&---
1 & 1 & 4 & 4 & 0
\end{array}
\Rightarrow x = 1 \text{ is a zero.}
\]

\( P(x) = x^3 + 3x^2 - 4 = (x - 1)(x^2 + 4x + 4) = (x - 1)(x + 2)^2 \)

Therefore, the zeros are \( x = -2, 1 \).

12. \( P(x) = x^3 - 7x^2 + 14x - 8 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \pm 8 \). \( P(x) \) has 3 variations in sign and hence 1 or 3 positive real zeros.

\( P(-x) = -x^3 - 7x^2 - 14x - 8 \) has 0 variations in sign and hence no negative real zeros.

\[
\begin{array}{c|cccc}
  & 1 & -7 & 14 & -8 \\
---&---&---&---&---
1 & 1 & -6 & 8 & 0
\end{array}
\Rightarrow x = 1 \text{ is a zero.}
\]

So

\( P(x) = x^3 - 7x^2 + 14x - 8 = (x - 1)(x^2 - 6x + 8) = (x - 1)(x - 2)(x - 4) \)

Therefore, the zeros are \( x = 1, 2, 4 \).

13. \( P(x) = x^3 - 3x - 2 \). The possible rational zeros are \( \pm 1, \pm 2 \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -x^3 + 3x - 2 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|ccc}
  & 1 & 0 & -3 & -2 \\
---&---&---&---&---
1 & 1 & -2 & 2 & 0
\end{array}
\Rightarrow x = 1 \text{ is not a zero.}
\]

\( P(x) = x^3 - 3x - 2 = (x - 2)(x^2 + 2x + 1) = (x - 2)(x + 1)^2 \). Therefore, the zeros are \( x = 2, -1 \).

14. \( P(x) = x^3 + 4x^2 - 3x - 18 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18 \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -x^3 + 4x^2 + 3x - 18 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|ccc}
  & 1 & 4 & -3 & -18 \\
---&---&---&---&---
1 & 5 & 2 & -16 & 0
\end{array}
\Rightarrow x = 2 \text{ is a zero.}
\]

\( P(x) = x^3 + 4x^2 - 3x - 18 = (x - 2)(x^2 + 6x + 9) = (x - 2)(x + 3)^2 \). Therefore, the zeros are \( x = -3, 2 \).

15. \( P(x) = x^3 - 6x^2 + 12x - 8 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \pm 8 \). \( P(x) \) has 3 variations in sign and hence 1 or 3 positive real zeros. \( P(-x) = -x^3 - 6x^2 - 12x - 8 \) has no variations in sign and hence 0 negative real zeros.

\[
\begin{array}{c|ccc}
  & 1 & -6 & 12 & -8 \\
---&---&---&---&---
1 & -5 & 7 & -1 & 0
\end{array}
\Rightarrow x = 1 \text{ is not a zero.}
\]

\( P(x) = x^3 - 6x^2 + 12x - 8 = (x - 2)(x^2 - 4x + 4) = (x - 2)^3 \). Therefore, the zero is \( x = 2 \).
16. \( P(x) = x^3 - x^2 - 8x + 12 \). The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±12. \( P(-x) = -x^3 - x^2 + 8x + 12 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{c|cccc}
1 & 1 & -1 & -8 & 12 \\
1 & 0 & -8 & 4 \\
\hline
1 & 0 & -8 & 4
\end{array}
\]

\( P(x) = x^3 - x^2 - 8x + 12 = (x - 2) (x^2 + x - 6) = (x - 2) (x + 3) (x - 2) \). Therefore, the zeros are \( x = -3, 2 \).

17. \( P(x) = x^3 - 4x^2 + x + 6 \). The possible rational zeros are ±1, ±2, ±3, ±6. \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = -x^3 - 4x^2 - x + 6 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{c|cccc}
-1 & 1 & -4 & 1 & 6 \\
-1 & 5 & -6 \\
\hline
1 & -5 & 6 & 0
\end{array}
\Rightarrow x + 1 \text{ is a factor.}
\]

So

\( P(x) = x^3 - 4x^2 + x + 6 = (x + 1) (x^2 - 5x + 6) = (x + 1) (x - 3) (x - 2) \)

Therefore, the zeros are \( x = -1, 2, 3 \).

18. \( P(x) = x^3 - 4x^2 - 7x + 10 \). The possible rational zeros are ±1, ±2, ±5, ±10. \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = -x^3 - 4x^2 + 7x + 10 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{c|cccc}
1 & 1 & -4 & -7 & 10 \\
1 & -3 & -10 \\
\hline
1 & -3 & -10 & 0
\end{array}
\Rightarrow x = 1 \text{ is a zero.}
\]

So

\( P(x) = x^3 - 4x^2 - 7x + 10 = (x - 1) (x^2 - 3x - 10) = (x - 1) (x - 5) (x + 2) \)

Therefore, the zeros are \( x = -2, 1, 5 \).

19. \( P(x) = x^3 + 3x^2 + 6x + 4 \). The possible rational zeros are ±1, ±2, ±4. \( P(x) \) has no variation in sign and hence no positive real zeros. \( P(-x) = -x^3 + 3x^2 - 6x + 4 \) has 3 variations in sign and hence 1 or 3 negative real zeros.

\[
\begin{array}{c|cccc}
-1 & 1 & 3 & 6 & 4 \\
-1 & 2 & 4 & 0 \\
\hline
1 & 2 & 4 & 0
\end{array}
\Rightarrow x + 1 \text{ is a factor.}
\]

So \( P(x) = x^3 + 3x^2 + 6x + 4 = (x + 1) (x^2 + 2x + 4) \). Now, \( Q(x) = x^2 + 2x + 4 \) has no real zeros, since the discriminant of this quadratic is \( b^2 - 4ac = (2)^2 - 4(1)(4) = -12 < 0 \). Thus, the only real zero is \( x = -1 \).

20. \( P(x) = x^3 - 2x^2 - 2x - 3 \). The possible rational zeros are ±1, ±3. \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -x^3 - 2x^2 + 2x - 3 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & -2 & -2 & -3 \\
1 & -1 & -3 \\
\hline
1 & -1 & -3 & -6
\end{array}
\quad
\begin{array}{c|ccc}
3 & 1 & -2 & -2 & -3 \\
3 & 3 & 3 \\
\hline
1 & 1 & 1 & 0
\end{array}
\Rightarrow x = 3 \text{ is a zero.}
\]

So \( P(x) = x^3 - 2x^2 - 2x - 3 = (x - 3) (x^2 + x + 1) \). Now, \( Q(x) = x^2 + x + 1 \) has no real zeros, since the discriminant is \( b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0 \). Thus, the only real zero is \( x = 3 \).
21. Method 1: \( P(x) = x^4 - 5x^2 + 4 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4 \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = x^4 - 5x^2 + 4 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{ccccc}
1 & 0 & -5 & 0 & 4 \\
1 & 1 & -4 & -4 & 0 \\
\end{array}
\]

\( x = 1 \) is a zero.

Thus \( P(x) = x^4 - 5x^2 + 4 = (x - 1)(x^3 + x^2 - 4x - 4) \). Continuing with the quotient we have:

\[
\begin{array}{cccc}
-1 & 1 & 1 & -4 & 4 \\
-1 & 0 & 4 & 0 \\
\end{array}
\]

\( x = -1 \) is a zero.

\[
P(x) = x^4 - 5x^2 + 4 = (x - 1)(x + 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2).
\]

Therefore, the zeros are \( x = \pm 1, \pm 2 \).

Method 2: Substituting \( u = x^2 \), the polynomial becomes \( P(u) = u^2 - 5u + 4 \), which factors:

\[
u^2 - 5u + 4 = (u - 1)(u - 4) = (x^2 - 1)(x^2 - 4),
\]

so either \( x^2 = 1 \) or \( x^2 = 4 \). If \( x^2 = 1 \), then \( x = \pm 1 \); if \( x^2 = 4 \), then \( x = \pm 2 \). Therefore, the zeros are \( x = \pm 1, \pm 2 \).

22. \( P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4 \). Using synthetic division, we see that \( (x - 1) \) is a factor of \( P(x) \):

\[
\begin{array}{cccc}
1 & -2 & -3 & 8 & -4 \\
1 & -1 & -4 & 4 & 0 \\
\end{array}
\]

\( x = 1 \) is a zero.

We continue by factoring the quotient, and we see that \( (x - 1) \) is again a factor:

\[
\begin{array}{cccc}
1 & -1 & -4 & 4 \\
1 & 0 & -4 & 0 \\
\end{array}
\]

\( x = 1 \) is a zero.

\[
P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4 = (x - 1)(x - 1)(x^2 - 4) = (x - 1)^2(x - 2)(x + 2)
\]

Therefore, the zeros are \( x = 1, \pm 2 \).

23. \( P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \pm 8 \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = x^4 - 6x^3 + 7x^2 + 6x - 8 \) has 3 variations in sign and hence 1 or 3 negative real zeros.

\[
\begin{array}{cccc}
1 & 1 & 6 & 7 & -6 & -8 \\
1 & 7 & 14 & 8 & 0 \\
\end{array}
\]

\( x = 1 \) is a zero.

and there are no other positive zeros. Thus \( P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8 = (x - 1)(x^3 + 7x^2 + 14x + 8) \).

Continuing by factoring the quotient, we have:

\[
\begin{array}{cccc}
-1 & 1 & 7 & 14 & 8 \\
-1 & -6 & -8 & 0 \\
\end{array}
\]

\( x = -1 \) is a zero.

So \( P(x) = x^4 + 6x^3 + 7x^2 - 6x - 8 = (x - 1)(x + 1)(x^2 + 6x + 8) = (x - 1)(x + 1)(x + 2)(x + 4) \). Therefore, the zeros are \( x = -4, -2, \pm 1 \).
4. \( P(x) = x^4 - x^3 - 23x^2 - 3x + 90 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 9, \pm 10, \pm 15, \pm 18, \pm 30, \pm 45, \pm 90 \). Since \( P(x) \) has 2 variations in sign, \( P \) has 0 or 2 positive real zeros. Since \( P(-x) = x^4 + x^3 - 23x^2 + 3x + 90 \) has 2 variations in sign, \( P \) has 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
 & 1 & -1 & -23 & -3  \\
1 & 1 & 0 & -23 & -26  \\
\hline
 & 1 & 0 & -23 & -26  \\
\end{array}
\]

\( P(x) = (x - 2) (x^3 + x^2 - 21x - 45) \). Continuing with the quotient we have:

\[
\begin{array}{c|cccc}
3 & 1 & 1 & -21 & -45  \\
\hline
 & 3 & 12 & -9 & -72  \\
\end{array}
\]

\( x = 2 \) is a zero. 

\[
\begin{array}{c|cccc}
 & 1 & 1 & -23 & -3  \\
2 & 2 & -42 & -90  \\
\hline
 & 2 & -21 & -45 & 0  \\
\end{array}
\]

\( \Rightarrow x = 2 \) is a zero.

5. \( P(x) = 4x^4 - 25x^2 + 36 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36, \pm 5, \pm 3\sqrt{2}, \pm 3\sqrt{2}, \pm \frac{9}{2}, \pm \frac{9}{2} \). Since \( P(x) \) has 2 variations in sign, there are 0 or 2 positive real zeros. Since \( P(-x) = 4x^4 - 25x^2 + 36 \) has 2 variations in sign, there are 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
 & 4 & 0 & -25 & 0  \\
1 & 4 & 0 & -25 & 0  \\
\hline
 & 4 & 4 & -21 & -21  \\
\end{array}
\]

\( x = 2 \) is an upper bound.

\[
\begin{array}{c|cccc}
 & 1 & 8 & -9 & -18  \\
\hline
 & 8 & 32 & 46  \\
\end{array}
\]

\( \Rightarrow \) all positive, \( x = 2 \) is an upper bound.

\[
\begin{array}{c|cccc}
 & 1 & 8 & -9 & -18  \\
\hline
 & 6 & 12 & 0  \\
\end{array}
\]

\( \Rightarrow x = \frac{3}{2} \) is a zero.

\[
\begin{array}{c|cccc}
 & 1 & 1 & -5 & 3 & 6  \\
2 & 2 & 0 & -5 & -2 & 4  \\
\hline
 & 2 & 2 & -6 & -6 &  \\
\end{array}
\]

\( \Rightarrow x = 2 \) is a zero.

Note: Since \( P(x) \) has only even terms, factoring by substitution also works. Let \( x^2 = u \); then 

\[ P(u) = 4u^2 - 25u + 36 = (u - 4) (4u - 9) = (x^2 - 4) (4x^2 - 9), \]

which gives the same results.

25. \( P(x) = 4x^4 - 25x^2 + 36 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36, \pm 5, \pm 3\sqrt{2}, \pm 3\sqrt{2}, \pm \frac{9}{2}, \pm \frac{9}{2} \). Since \( P(x) \) has 2 variations in sign, there are 0 or 2 positive real zeros. Since \( P(-x) = 4x^4 - 25x^2 + 36 \) has 2 variations in sign, there are 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
 & 4 & 0 & -25 & 0  \\
1 & 4 & 0 & -25 & 0  \\
\hline
 & 4 & 4 & -21 & -21  \\
\end{array}
\]

\( x = 2 \) is an upper bound.

\[
\begin{array}{c|cccc}
 & 4 & 8 & -9 & -18  \\
\hline
 & 8 & 32 & 46  \\
\end{array}
\]

\( \Rightarrow \) all positive, \( x = 2 \) is an upper bound.

\[
\begin{array}{c|cccc}
 & 4 & 8 & -9 & -18  \\
\hline
 & 6 & 12 & 0  \\
\end{array}
\]

\( \Rightarrow x = \frac{3}{2} \) is a zero.

\( P(x) = (x - 2) (x - 3) (2x^2 + 7x + 6) = (x - 2) (x - 3) (2x + 3) (x + 2). \) Therefore, the zeros are \( x = \pm 2, \pm \frac{3}{2}. \)

26. \( P(x) = x^4 - x^3 - 5x^2 + 3x + 6 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6 \). Since \( P(x) \) has 2 variations in sign, \( P \) has 0 or 2 positive real zeros. Since \( P(-x) = x^4 + x^3 - 5x^2 - 3x + 6 \) has 2 variations in sign, \( P \) has 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
 & 1 & -1 & -5 & 3 & 6  \\
1 & 1 & 0 & -5 & -2  \\
\hline
 & 1 & 0 & -5 & -2  \\
\end{array}
\]

\( P(x) = (x - 2) (x^3 + x^2 - 3x - 3) \). Continuing with the quotient we have:

\[
\begin{array}{c|cccc}
 & 1 & 1 & -3 & -3  \\
3 & 3 & 12 & 27  \\
\hline
 & 1 & 4 & 9 & 24  \\
\end{array}
\]

\( \Rightarrow x = 3 \) is an upper bound.

\[
\begin{array}{c|cccc}
 & 1 & -1 & -5 & 3 & 6  \\
2 & 2 & -6 & -6  \\
\hline
 & 1 & 1 & -3 & -3 & 0  \\
\end{array}
\]

\( \Rightarrow x = 2 \) is a zero.

\[
\begin{array}{c|cccc}
 & 1 & 1 & -3 & -3  \\
3 & 3 & -1 & 0 & 3  \\
\hline
 & 1 & 0 & -3 & 0  \\
\end{array}
\]

\( \Rightarrow x = -1 \) is a zero.

\( P(x) = (x - 2) (x + 1) (x^2 - 3) \). Therefore, the rational zeros are \( x = -1, 2 \).
27. \( P(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 \). The possible rational zeros are ±1, ±2, ±4, ±8, ±16. \( P(x) \) has no variations in sign and hence no positive real zero. \( P(-x) = x^4 - 8x^3 + 24x^2 - 32x + 16 \) has 4 variations in sign and hence 0 or 2 or negative real zeros.

\[
\begin{array}{c|cccc}
-1 & 1 & 8 & 24 & 32 & 16 \\
& -1 & -7 & -17 & -15 \\
\hline
1 & 7 & 17 & 15 & 1
\end{array}
\Rightarrow x = -1 \text{ is not a zero.}
\]

Thus \( P(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 = (x + 2)(x^3 + 6x^2 + 12x + 8) \). Continuing by factoring the quotient, we have

\[
\begin{array}{c|cccc}
-2 & 1 & 6 & 12 & 8 \\
& -2 & -8 & -8 \\
\hline
1 & 4 & 4 & 0
\end{array}
\Rightarrow x = -2 \text{ is a zero.}
\]

Thus \( P(x) = (x + 2)^2(x^2 + 4x + 4) = (x + 2)^4 \). Therefore, the zero is \( x = -2 \).

28. \( P(x) = 2x^3 + 7x^2 + 4x - 4 \). The possible rational zeros are ±1, ±2, ±4, ±\( \frac{1}{2} \). Since \( P(x) \) has 1 variation in sign, \( P \) has 1 positive real zero. Since \( P(-x) = -2x^3 + 7x^2 - 4x - 4 \) has 2 variations in sign, \( P \) has 0 or 2 negative real zeros.

\[
\begin{array}{c|ccc}
1 & 2 & 7 & 4 & -4 \\
& 2 & 9 & 13 \\
\hline
2 & 9 & 13 & 9
\end{array}
\Rightarrow x = 1 \text{ is an upper bound.}
\]

\[
\begin{array}{c|ccc}
\frac{1}{2} & 2 & 7 & 4 & -4 \\
& 1 & 4 & 4 \\
\hline
2 & 8 & 8 & 0
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

\( P(x) = (x - \frac{1}{2})(2x^2 + 8x + 8) = 2\left(x - \frac{1}{2}\right)(x^2 + 4x + 4) = 2\left(x - \frac{1}{2}\right)(x + 2)^2 \). Therefore, the zeros are \( x = -2, \frac{1}{2} \).

29. Factoring by grouping can be applied to this exercise. \( 4x^3 + 4x^2 - x - 1 = 4x^2(x + 1) - (x + 1) = (x + 1)(4x^2 - 1) = (x + 1)(2x + 1)(2x - 1) \). Therefore, the zeros are \( x = -1, \pm \frac{1}{2} \).

30. We use factoring by grouping: \( P(x) = 2x^3 - 3x^2 - 2x + 3 = 2x(x^2 - 1) - 3(x^2 - 1) = (x - 1)(x + 1)(2x - 3) \). Therefore, the zeros are \( x = \frac{3}{2}, \pm 1 \).

31. \( P(x) = 4x^3 - 7x + 3 \). The possible rational zeros are ±1, ±3, ±\( \frac{1}{2} \), ±\( \frac{3}{2} \), ±\( \frac{1}{4} \), ±\( \frac{3}{4} \). Since \( P(x) \) has 2 variations in sign, there are 0 or 2 positive zeros. Since \( P(-x) = -4x^3 + 7x + 3 \) has 1 variation in sign, there is 1 negative zero.

\[
\begin{array}{c|ccc}
\frac{1}{2} & 4 & 0 & -7 & 3 \\
& 2 & 1 & -3 \\
\hline
4 & 2 & -6 & 0
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

\( P(x) = (x - \frac{1}{2})(4x^2 + 2x - 6) = (2x - 1)(2x^2 + x - 3) = (2x - 1)(x - 1)(2x + 3) = 0 \). Thus, the zeros are \( x = -\frac{3}{2}, \frac{1}{2}, 1 \).

32. \( P(x) = 8x^3 + 10x^2 - x - 3 \). The possible rational zeros are ±1, ±3, ±\( \frac{1}{2} \), ±\( \frac{3}{2} \), ±\( \frac{1}{4} \), ±\( \frac{3}{4} \), ±\( \frac{1}{8} \), ±\( \frac{3}{8} \). Since \( P(x) \) has 1 variation in sign, \( P \) has 1 positive real zero. Since \( P(-x) = -8x^3 + 10x^2 + x - 3 \) has 2 variations in sign, \( P \) has 0 or 2 negative real zeros.

\[
\begin{array}{c|ccc}
1 & 8 & 10 & -1 & -3 \\
& 8 & 18 & 17 \\
\hline
8 & 18 & 17 & 14
\end{array}
\Rightarrow x = 1 \text{ is an upper bound.}
\]

\[
\begin{array}{c|ccc}
\frac{1}{2} & 8 & 10 & -1 & -3 \\
& 4 & 7 & 3 \\
\hline
8 & 14 & 6 & 0
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

\( P(x) = 8x^3 + 10x^2 - x - 3 = (x - \frac{1}{2})(8x^2 + 14x + 6) = 2\left(x - \frac{1}{2}\right)(4x^2 + 7x + 3) = (2x - 1)(x + 1)(4x + 3) = 0 \)

Therefore, the zeros are \( x = -1, -\frac{3}{4}, \frac{1}{2} \).
SECTION 3.3 Real Zeros of Polynomials 233

1. \( P(x) = 4x^3 + 8x^2 - 11x - 15 \). The possible rational zeros are \( \pm 1, \pm 3, \pm 5, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{5}{4} \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -4x^3 + 8x^2 + 11x - 15 \) has 2 variations in sign, so \( P \) has 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 4 & 8 & -11 & -15 \\
\hline
\multicolumn{1}{|c|}{} & 4 & 12 & 1 & -14 \\
\end{array}
\Rightarrow x = 1 \text{ is not a zero.}
\]

\[
\begin{array}{c|cccc}
5 & 4 & 8 & -11 & -15 \\
\hline
\multicolumn{1}{|c|}{} & 20 & 140 & 645 & \\
\end{array}
\Rightarrow x = 5 \text{ is not a zero.}
\]

\[
\begin{array}{c|cccc}
\frac{1}{2} & 4 & 8 & -11 & -15 \\
\hline
\multicolumn{1}{|c|}{} & 2 & 5 & -3 & \\
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is not a zero.}
\]

\[
\begin{array}{c|cccc}
\frac{3}{4} & 4 & 8 & -11 & -15 \\
\hline
\multicolumn{1}{|c|}{} & 4 & 14 & 10 & 0 \\
\end{array}
\Rightarrow x = \frac{3}{4} \text{ is a zero.}
\]

Thus \( P(x) = 4x^3 + 8x^2 - 11x - 15 = (x - \frac{3}{4}) (4x^2 + 14x + 10) \). Continuing by factoring the quotient, whose possible rational zeros are \(-1, -5, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \text{ and } -\frac{5}{4} \), we have

\[
\begin{array}{c|cccc}
-1 & 4 & 14 & 10 \\
\hline
\multicolumn{1}{|c|}{} & -4 & -10 & \\
\end{array}
\Rightarrow x = -1 \text{ is a zero.}
\]

Thus \( P(x) = (x - \frac{3}{4}) (x + 1) (4x + 10) \) has zeros \( \frac{3}{4}, -1, \text{ and } -\frac{5}{4} \).

4. \( P(x) = 6x^3 + 11x^2 - 3x - 2 \). The possible rational zeros are \( \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{1}{2}, \pm \frac{2}{3} \). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -6x^3 + 11x^2 + 3x - 2 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 6 & 11 & -3 & -2 \\
\hline
\multicolumn{1}{|c|}{} & 6 & 17 & 14 & \\
\end{array}
\Rightarrow x = 1 \text{ is not a zero.}
\]

\[
\begin{array}{c|cccc}
\frac{1}{2} & 6 & 11 & -3 & -2 \\
\hline
\multicolumn{1}{|c|}{} & 6 & 14 & 4 & 0 \\
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

Thus, \( P(x) = 6x^3 + 11x^2 - 3x - 2 = (x - \frac{1}{2}) (6x^2 + 14x + 4) \). Continuing:

\[
\begin{array}{c|cccc}
-1 & 6 & 14 & 4 \\
\hline
\multicolumn{1}{|c|}{} & -6 & -8 & \\
\end{array}
\Rightarrow x = -1 \text{ is not a zero.}
\]

\[
\begin{array}{c|cccc}
-2 & 6 & 14 & 4 \\
\hline
\multicolumn{1}{|c|}{} & -12 & -4 & \\
\end{array}
\Rightarrow x = -2 \text{ is a zero.}
\]

Thus, \( P(x) = (x - \frac{1}{2}) (x + 2) (6x + 2) \) has zeros \( \frac{1}{2}, -2, \text{ and } -\frac{1}{3} \).
35. \( P(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4 \). The possible rational zeros are ±1, ±2, ±4, ±\( \frac{1}{2} \). \( P(x) \) has 3 variations in sign and hence 1 or 3 positive real zeros. \( P(-x) = 2x^4 + 7x^3 + 3x^2 - 8x - 4 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{c|cccc}
1 & 2 & -7 & 3 & 8 & -4 \\
\hline
& 2 & -5 & -2 & 6 \\
2 & -5 & -2 & 6 & 2 \\
\Rightarrow x = 1 \text{ is not a zero.}
\end{array}
\]

Thus \( P(x) = 2x^4 - 7x^3 + 3x^2 + 8x - 4 = (x - \frac{1}{2}) (2x^3 - 6x^2 + 8) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{c|cccc}
2 & 2 & -6 & 0 & 8 \\
\hline
& 4 & -4 & -8 \\
2 & -2 & -4 & 0 \\
\Rightarrow x = 2 \text{ is a zero.}
\end{array}
\]

\[ P(x) = (x - \frac{1}{2}) (x - 2) (2x^2 - 2x - 4) = 2 (x - \frac{1}{2})(x - 2) (x^2 - x - 2) = 2 (x - \frac{1}{2})(x - 2)^2 (x + 1). \]

Thus, the zeros are \( x = \frac{1}{2}, 2, -1 \).

36. \( P(x) = 6x^4 - 7x^3 - 12x^2 + 3x + 2 \). The possible rational zeros are ±1, ±2, ±\( \frac{1}{2} \), ±\( \frac{1}{3} \), ±\( \frac{1}{4} \), ±\( \frac{1}{6} \). Since \( P(x) \) has 2 variations in sign, \( P \) has 0 or 2 positive real zeros. Since \( P(-x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 \) has 2 variations in sign, has 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 6 & -7 & -12 & 3 & 2 \\
\hline
& 6 & -1 & -13 & -10 \\
6 & -1 & -13 & -10 & 8 \\
\Rightarrow x = 2 \text{ is a zero.}
\end{array}
\]

\[ P(x) = 6x^4 - 7x^3 - 12x^2 + 3x + 2 = (x - 2) (6x^3 + 5x^2 - 2x - 1). \]

Continuing by factoring the quotient, first note that the possible rational zeros are -1, ±\( \frac{1}{2} \), ±\( \frac{1}{3} \), ±\( \frac{1}{6} \). We have:

\[
\begin{array}{c|cccc}
\frac{1}{2} & 6 & 5 & -2 & -1 \\
\hline
& 3 & 4 & 1 \\
6 & 8 & 2 & 0 \\
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\end{array}
\]

\[ P(x) = (x - 2) (x - \frac{1}{2}) (6x^2 + 8x + 2) = 2 (x - 2) (x - \frac{1}{2}) (3x^2 + 4x + 1) = (x - 2) (x - \frac{1}{2}) (x + 1) (3x + 1). \]

Therefore, the zeros are \( x = -1, -\frac{1}{2}, \frac{1}{3}, 2 \).

37. \( P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36 \). The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±8, ±9, ±12, ±18. \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = -x^5 + 3x^4 + 9x^3 - 31x^2 + 36 \) has 3 variations in sign and hence 1 or 3 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & 3 & -9 & -31 & 0 & 36 \\
\hline
& 1 & 4 & -5 & -36 & -36 \\
1 & 4 & -5 & -36 & -36 & 0 \\
\Rightarrow x = 1 \text{ is a zero.}
\end{array}
\]

So \( P(x) = x^5 + 3x^4 - 9x^3 - 31x^2 + 36 = (x - 1) (x^4 + 4x^3 - 5x^2 - 36x - 36) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{c|cccc}
1 & 1 & 4 & -5 & -36 & -36 \\
\hline
& 1 & 5 & 0 & -36 \\
1 & 1 & 0 & -36 & -72 \\
\Rightarrow x = 3 \text{ is a zero.}
\end{array}
\]

\[
\begin{array}{c|cccc}
2 & 1 & 4 & -5 & -36 & -36 \\
\hline
& 2 & 12 & 14 & -44 \\
1 & 6 & 7 & -22 & -80 \\
\Rightarrow x = 3 \text{ is a zero.}
\end{array}
\]

\[
\begin{array}{c|cccc}
3 & 1 & 4 & -5 & -36 & -36 \\
\hline
& 3 & 21 & 48 & 36 \\
1 & 7 & 16 & 12 & 0 \\
\Rightarrow x = 3 \text{ is a zero.}
\end{array}
\]
So \( P(x) = (x-1)(x-3)(x^3 + 7x^2 + 16x + 12) \). Since we have 2 positive zeros, there are no more positive zeros, so we continue by factoring the quotient with possible negative zeros.

\[
\begin{array}{cccccc}
-1 & | & 1 & 7 & 16 & 12 \\
& & -1 & -6 & -10 & 2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
-2 & | & 1 & 7 & 16 & 12 \\
& & -2 & -10 & -12 & 0 \\
\end{array}
\Rightarrow x = -2 \text{ is a zero.}
\]

Then \( P(x) = (x-1)(x-3)(x+2)(x^2 + 5x + 6) = (x-1)(x-3)(x+2)^2(x+3) \). Thus, the zeros are \( x = 1, 3, -2, -3 \).

38. \( P(x) = x^5 - 4x^4 - 3x^3 + 22x^2 - 4x - 24 \) has possible rational zeros \( \pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm24 \). Since \( P(x) \) has 3 variations in sign, there are 1 or 3 positive real zeros. Since \( P(-x) = -x^5 - 4x^4 + 3x^3 + 22x^2 + 4x - 24 \) has 2 variations in sign, there are 0 or 2 negative real zeros.

\[
\begin{array}{cccccc}
1 & | & 1 & -4 & -3 & 22 & -4 & -24 \\
& & 1 & -3 & -6 & 16 & 12 \\
\end{array}
\]

\[
\begin{array}{cccccc}
2 & | & 1 & -4 & -3 & 22 & -4 & -24 \\
& & 2 & -4 & -14 & 16 & 24 \\
\end{array}
\Rightarrow x = 2 \text{ is a zero.}
\]

\( P(x) = (x-2)(x^4 - 2x^3 - 7x^2 + 8x + 12) \)

\[
\begin{array}{cccc}
2 & | & 1 & -2 & -7 & 8 & 12 \\
& & 2 & 0 & -14 & -12 & 2 \\
\end{array}
\Rightarrow x = 2 \text{ is a zero again.}
\]

\( P(x) = (x-2)^2(x^3 - 7x - 6) \)

\[
\begin{array}{cccc}
3 & | & 1 & 0 & -7 & -6 \\
& & 3 & 9 & 6 & 3 \\
\end{array}
\Rightarrow x = 3 \text{ is a zero.}
\]

\( P(x) = (x-2)^2(x-3)(x^2 + 3x + 2) = (x-2)^2(x-3)(x+1)(x+2) = 0 \). Therefore, the zeros are \( x = -1, 2, 3 \).

39. \( P(x) = 3x^5 - 14x^4 - 14x^3 + 36x^2 + 43x + 10 \) has possible rational zeros \( \pm1, \pm2, \pm5, \pm10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3} \). Since \( P(x) \) has 2 variations in sign, there are 0 or 2 positive real zeros. Since \( P(-x) = -3x^5 - 14x^4 + 14x^3 + 36x^2 - 43x + 10 \) has 3 variations in sign, there are 1 or 3 negative real zeros.

\[
\begin{array}{cccccc}
1 & | & 3 & -14 & -14 & 36 & 43 & 10 \\
& & 3 & -11 & -25 & 11 & 54 & 64 \\
\end{array}
\]

\[
\begin{array}{cccccc}
2 & | & 3 & -14 & -14 & 36 & 43 & 10 \\
& & 6 & -16 & -60 & -48 & -10 & 0 \\
\end{array}
\Rightarrow x = 2 \text{ is a zero.}
\]

\( P(x) = (x-2)(3x^4 - 8x^3 - 30x^2 - 24x - 5) \)

\[
\begin{array}{cccccc}
2 & | & 3 & -8 & -30 & -24 & -5 \\
& & 6 & -4 & -68 & -184 \\
\end{array}
\]

\[
\begin{array}{cccccc}
3 & | & 3 & -8 & -30 & -24 & -5 \\
& & 15 & 35 & 25 & 5 \\
\end{array}
\Rightarrow x = 5 \text{ is a zero.}
\]

\( P(x) = (x-2)(x-5)(3x^3 + 7x^2 + 5x + 1) \). Since \( 3x^3 + 7x^2 + 5x + 1 \) has no variation in sign, there are no more positive zeros.

\[
\begin{array}{cccc}
-1 & | & 3 & 7 & 5 & 1 \\
& & -3 & -4 & -1 \\
\end{array}
\Rightarrow x = -1 \text{ is a zero.}
\]

\( P(x) = (x-2)(x-5)(x+1)(3x^2 + 4x + 1) = (x-2)(x-5)(x+1)(3x+1) \). Therefore, the zeros are \( x = -1, -\frac{1}{3}, 2, 5 \).
40. \( P(x) = 2x^6 - 3x^5 - 13x^4 + 29x^3 - 27x^2 + 32x - 12 \) has possible rational zeros ±1, ±2, ±3, ±4, ±6, ±12, ±\(\frac{1}{2}\), ±\(\frac{3}{2}\). Since \( P(x) \) has 5 variations in sign, there are 1 or 3 or 5 positive real zeros. Since \( P(-x) = 2x^6 + 3x^5 - 13x^4 - 29x^3 + 27x^2 - 32x - 12 \) has 3 variations in sign, there are 1 or 3 negative real zeros.

\[
\begin{array}{c|cccccc}
1 & 2 & -3 & -13 & 29 & -27 & 32 & -12 \\
 2 & 1 & -14 & 15 & -12 & 20 & 8 \\
 4 & 2 & -22 & 14 & -26 & 12 \\
 2 & 1 & -11 & 7 & -13 & 6 & 0 \\
 4 & 2 & -22 & 14 & -26 & 12 \\
\end{array}
\]

\( P(x) = (x - 2) (2x^5 + x^4 - 11x^3 + 7x^2 - 13x + 6). \) We continue with the quotient:

\[
\begin{array}{c|cccc}
2 & 1 & -11 & 7 & -13 \\
 4 & 10 & -2 & 10 & -6 \\
 2 & 5 & -1 & 5 & -3 \\
 1 & 2 & 1 & 3 \\
 5 & 10 & 6 & 0 \\
\end{array}
\]

\( P(x) = (x - 2) (2x^4 + 5x^3 - x^2 + 5x - 3). \) We continue with the quotient, first noting 2 is no longer a possible rational solution:

\[
\begin{array}{c|cccc}
3 & 2 & 5 & -1 & 5 \\
 6 & 22 & 42 & 94 \\
 2 & 11 & 21 & 47 & 91 \\
\end{array}
\]

We know that there is at least 1 more positive zero.

\[
\frac{1}{2} \begin{array}{c|cccc}
2 & 5 & -1 & 5 & -3 \\
 1 & 2 & 1 & 3 \\
 2 & 6 & 2 & 6 & 0 \\
\end{array}
\]

\( P(x) = (x - 2)^2 (x - \frac{1}{2}) (2x^3 + 6x^2 + 2x + 6). \) We can factor \( 2x^3 + 6x^2 + 2x + 6 = (2x^3 + 6x^2) + (2x + 6) = (2x + 6) (x^2 + 1). \) So \( P(x) = 2(x - 2)^2 (x - \frac{1}{2}) (x + 3) (x^2 + 1). \)

Since \( x^2 + 1 \) has no real zeros, the zeros of \( P \) are \( x = -3, 2, \frac{1}{2} \).

41. \( P(x) = x^3 + 4x^2 + 3x - 2. \) The possible rational zeros are ±1, ±2. \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -x^3 + 4x^2 - 3x - 2 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & 4 & 3 & -2 \\
 1 & 5 & 8 \\
\end{array}
\Rightarrow x = 1 \text{ is an upper bound.}
\]

\[
\begin{array}{c|cccc}
-2 & 1 & 4 & 3 & -2 \\
 -2 & -4 & 2 \\
 1 & 2 & -1 & 0 \\
\end{array}
\Rightarrow x = -2 \text{ is a zero.}
\]

So \( P(x) = (x + 2) (x^2 + 2x - 1). \) Using the quadratic formula on the second factor, we have:

\[
x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}. \text{ Therefore, the zeros are } x = -2, -1 + \sqrt{2}, -1 - \sqrt{2}.\]
2. \( P(x) = x^3 - 5x^2 + 2x + 12 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \). \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = -x^3 - 5x^2 - 2x + 12 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{c|cccc}
1 & 1 & -5 & 2 & 12 \\
1 & 1 & -4 & -2 & 12 \\
3 & 1 & -5 & 2 & 12 \\
3 & 1 & -6 & -12 & 0 \\
1 & 1 & -2 & -4 & 0 \\
\end{array}
\Rightarrow x = 3 \text{ is a zero.}
\]

So \( P(x) = (x - 3) (x^2 - 2x - 4) \). Using the quadratic formula on the second factor, we have:

\[
x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)} = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}.
\]

Therefore, the zeros are \( x = 3, 1 \pm \sqrt{5} \).

13. \( P(x) = x^4 - 6x^3 + 4x^2 + 15x + 4 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4 \). \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = x^4 + 6x^3 + 4x^2 - 15x + 4 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & -6 & 4 & 15 & 4 \\
1 & 1 & -5 & -1 & 14 \\
4 & 1 & -6 & 4 & 15 \\
4 & 1 & -2 & -4 & -1 \\
\end{array}
\Rightarrow x = 4 \text{ is a zero.}
\]

So \( P(x) = (x - 4) (x^3 - 2x^2 - 4x - 1) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{c|cccc}
4 & 1 & -2 & -4 & -1 \\
4 & 1 & 2 & 4 & 16 \\
\end{array}
\Rightarrow x = 4 \text{ is an upper bound.}
\]

\[
\begin{array}{c|cccc}
1 & 1 & -2 & -4 & -1 \\
1 & 1 & -3 & -1 & 0 \\
\end{array}
\Rightarrow x = -1 \text{ is a zero.}
\]

So \( P(x) = (x - 4) (x + 1) (x^2 - 3x - 1) \). Using the quadratic formula on the third factor, we have:

\[
x = \frac{(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \frac{3 \pm \sqrt{13}}{2}.
\]

Therefore, the zeros are \( x = 4, -1, \frac{3 \pm \sqrt{13}}{2} \).

44. \( P(x) = x^4 + 2x^3 - 2x^2 - 3x + 2 \). The possible rational zeros are \( \pm 1, \pm 2 \). \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = x^4 - 2x^3 - 2x^2 + 3x + 2 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & 2 & -2 & -3 & 2 \\
1 & 1 & 3 & 1 & -2 \\
\end{array}
\Rightarrow x = 1 \text{ is a zero.}
\]

\( P(x) = (x - 1) (x^3 + 3x^2 + x - 2) \). Continuing with the quotient:

\[
\begin{array}{c|cccc}
1 & 1 & 3 & 1 & -2 \\
1 & 1 & 4 & 5 & 3 \\
\end{array}
\Rightarrow x = 1 \text{ is an upper bound.}
\]

\[
\begin{array}{c|cccc}
1 & 1 & 3 & 1 & -2 \\
1 & 2 & -1 & -1 \\
\end{array}
\Rightarrow x = -2 \text{ is a zero.}
\]

So \( P(x) = (x - 1)(x + 2)(x^2 + x - 1) \). Using the quadratic formula on the third factor, we have:

\[
x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.
\]

Therefore, the zeros are \( x = 1, -2, \frac{1 \pm \sqrt{5}}{2} \).
45. \( P(x) = x^4 - 7x^3 + 14x^2 - 3x - 9 \). The possible rational zeros are \( \pm 1, \pm 3, \pm 9 \). \( P(x) \) has 3 variations in sign and hence 1 or 3 positive real zeros. \( P(-x) = x^4 + 7x^3 + 14x^2 + 3x - 4 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{rrrrr}
1 & 1 & -7 & 14 & -3 & -9 \\
1 & -6 & 8 & 5 &  \\
\hline
1 & -6 & 8 & 5 & 4
\end{array}
\quad
\begin{array}{rrrr}
3 & 1 & -7 & 14 & -3 & -9 \\
3 & -12 & 6 & 9 &  \\
\hline
3 & -12 & 6 & 9 & 0
\end{array}
\]

\( \Rightarrow x = 3 \) is a zero. So \( P(x) = (x - 3)(x^3 - 4x^2 + 2x + 3) \). Since the constant term of the second term is 3, \( \pm 9 \) are no longer possible zeros. Continuing by factoring the quotient, we have:

\[
\begin{array}{rrr}
3 & 1 & -4 & 2 & 3 \\
3 & -3 & -3 & 0 &  \\
\hline
3 & -3 & -3 & 0 & 0
\end{array}
\Rightarrow x = 3 \) is a zero again.

So \( P(x) = (x - 3)^2(x^2 - x - 1) \). Using the quadratic formula on the second factor, we have:

\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{2}.\]

Therefore, the zeros are \( x = 3, \frac{1 \pm \sqrt{5}}{2} \).

46. \( P(x) = x^5 - 4x^4 - x^3 + 10x^2 + 2x - 4 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4 \). \( P(x) \) has 3 variations in sign and hence 1 or 3 positive real zeros. \( P(-x) = -x^5 - 4x^4 + x^3 + 10x^2 - 2x - 4 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{rrrrr}
1 & 1 & -4 & -1 & 10 & 2 & -4 \\
1 & -3 & -4 & 6 & 8 &  \\
\hline
1 & -3 & -4 & 6 & 8 & 4
\end{array}
\quad
\begin{array}{rrrrr}
2 & 1 & -4 & -1 & 10 & 2 & -4 \\
2 & -4 & -10 & 0 & 4 &  \\
\hline
2 & -4 & -10 & 0 & 4 & 0
\end{array}
\Rightarrow x = 2 \) is a zero. So \( P(x) = (x - 2)(x^4 - 2x^3 - 5x^2 + 2) \). Since the constant term of the second factor is 2, \( \pm 4 \) are no longer possible zeros. Continuing by factoring the quotient, we have:

\[
\begin{array}{rrrr}
2 & 1 & -2 & -5 & 0 & 2 \\
2 & -2 & -10 & -20 &  \\
\hline
2 & -2 & -10 & -20 & 0
\end{array}
\quad
\begin{array}{rrrr}
-1 & 1 & -2 & -5 & 0 & 2 \\
-1 & -3 & 2 & -2 &  \\
\hline
-1 & -3 & 2 & -2 & 0
\end{array}
\Rightarrow x = -1 \) is a zero again.

So \( P(x) = (x - 2)(x + 1)(x^3 - 3x^2 - 2x - 2) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{rrrrr}
-1 & 1 & -3 & -2 & 2 \\
-1 & 4 & -2 & 0 &  \\
\hline
-1 & 4 & -2 & 0 & 0
\end{array}
\Rightarrow x = -1 \) is a zero again.

So \( P(x) = (x - 2)(x + 1)^2(x^2 - 4x + 2) \). Using the quadratic formula on the second factor, we have:

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}.\]

Therefore, the zeros are \( x = -1, 2, 2 \pm \sqrt{2} \).

47. \( P(x) = 4x^3 - 6x^2 + 1 \). The possible rational zeros are \( \pm 1, \pm \frac{1}{2} \). \( P(x) \) has 2 variations in sign and hence 0 or 2 positive real zeros. \( P(-x) = -4x^3 - 6x^2 + 1 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{rrrr}
1 & 4 & -6 & 0 & 1 \\
4 & -2 & -2 &  \\
\hline
4 & -2 & -2 & -1
\end{array}
\quad
\begin{array}{rrrr}
\frac{1}{2} & 4 & -6 & 0 & 1 \\
2 & -2 & -1 &  \\
\hline
2 & -2 & -1 & 0
\end{array}
\Rightarrow x = \frac{1}{2} \) is a zero. So \( P(x) = (x - \frac{1}{2})(4x^2 - 4x - 2) \). Using the quadratic formula on the second factor, we have:

\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-2)}}{2(4)} = \frac{4 \pm \sqrt{48}}{8} = \frac{4 \pm 4\sqrt{3}}{8} = \frac{1 \pm \sqrt{3}}{2}.\]

Therefore, the zeros are \( x = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2} \).
3. \( P(x) = 3x^3 - 5x^2 - 8x - 2 \). The possible rational zeros are ±1, ±2, ±\(\frac{1}{3}\), ±\(\frac{2}{3}\). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = -3x^3 - 5x^2 + 8x - 2 \) has 2 variations in sign and hence 0 or 2 negative real zeros.

\[
\begin{array}{c|ccc}
1 & 3 & -5 & -8 & -2 \\
\hline
& 3 & -2 & -10 & \\
3 & -2 & -10 & -12 & \\
\end{array}
\quad
\begin{array}{c|ccc}
2 & 3 & -5 & -8 & -2 \\
\hline
& 6 & 2 & -12 & \\
3 & 1 & -6 & -14 & \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\frac{1}{3} & 3 & -5 & -8 & -2 \\
\hline
& 1 & -\frac{4}{3} & -\frac{28}{9} & \\
3 & -4 & -\frac{28}{9} & -\frac{46}{9} & \\
\end{array}
\quad
\begin{array}{c|ccc}
\frac{2}{3} & 3 & -5 & -8 & -2 \\
\hline
& 2 & -2 & -\frac{20}{3} & \\
3 & -3 & -10 & -\frac{20}{3} & \\
\end{array}
\]

Thus we have tried all the positive rational zeros, so we try the negative zeros.

\[
\begin{array}{c|ccc}
-1 & 3 & -5 & -8 & -2 \\
\hline
& -3 & 8 & 0 & \\
3 & -8 & 0 & -2 & \\
\end{array}
\quad
\begin{array}{c|ccc}
-2 & 3 & -5 & -8 & -2 \\
\hline
& -6 & 22 & -28 & \\
3 & -11 & 14 & -30 & \\
\end{array}
\]

\[
\begin{array}{c|ccc}
-\frac{1}{3} & 3 & -5 & -8 & -2 \\
\hline
& -1 & 2 & 2 & \\
3 & -6 & -6 & 0 & \\
\Rightarrow x = -\frac{1}{3} \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x + \frac{1}{3}) (3x^2 - 6x - 6) = 3 (x + \frac{1}{3}) (x^2 - 2x - 2) \). Using the quadratic formula on the second factor, we have:

\[
x = \frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-2)}}{2(3)} = \frac{2\pm\sqrt{16}}{6} = \frac{2\pm2\sqrt{3}}{6} = 1 \pm \sqrt{3}.
\]

Therefore, the zeros are \( x = -\frac{1}{3}, 1 \pm \sqrt{3} \).

19. \( P(x) = 2x^4 + 15x^3 + 17x^2 + 3x - 1 \). The possible rational zeros are ±1, ±\(\frac{1}{2}\). \( P(x) \) has 1 variation in sign and hence 1 positive real zero. \( P(-x) = 2x^4 - 15x^3 + 17x^2 - 3x - 1 \) has 3 variations in sign and hence 1 or 3 negative real zeros.

\[
\begin{array}{c|cccc}
\frac{1}{2} & 2 & 15 & 17 & 3 & -1 \\
\hline
& 1 & 8 & \frac{25}{2} & \frac{31}{4} & \\
2 & 16 & 25 & \frac{31}{2} & \frac{27}{4} & \\
\Rightarrow x = \frac{1}{2} \text{ is an upper bound.}
\end{array}
\]

\[
\begin{array}{c|cccc}
-\frac{1}{2} & 2 & 15 & 17 & 3 & -1 \\
\hline
& -1 & -7 & -5 & 1 & \\
2 & 14 & 10 & -2 & 0 & \\
\Rightarrow x = -\frac{1}{2} \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x + \frac{1}{2}) (2x^3 + 14x^2 + 10x - 2) = 2 (x + \frac{1}{2}) (x^3 + 7x^2 + 5x - 1) \).

\[
\begin{array}{c|ccc}
-1 & 1 & 7 & 5 & -1 \\
\hline
& -1 & -6 & 1 & \\
1 & 6 & -1 & 0 & \\
\Rightarrow x = -1 \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x + \frac{1}{2}) (2x^3 + 14x^2 + 10x - 2) = 2 (x + \frac{1}{2}) (x + 1) (x^2 + 6x - 1) \) Using the quadratic formula on the third factor, we have:

\[
x = \frac{-6\pm\sqrt{(-6)^2-4(2)(-1)}}{2(2)} = \frac{-6\pm\sqrt{10}}{4} = \frac{-3\pm\sqrt{10}}{2}.
\]

Therefore, the zeros are \( x = -1, -\frac{1}{2}, -3 \pm \sqrt{10} \).
50. \( P(x) = 4x^5 - 18x^4 - 6x^3 + 91x^2 - 60x + 9 \). The possible rational zeros are \( \pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2} \).

\( P(x) \) has 4 variations in sign and hence 0 or 2 or 4 positive real zeros. \( P(-x) = -4x^5 - 18x^4 + 6x^3 + 91x^2 + 60x + 9 \) has 1 variation in sign and hence 1 negative real zero.

\[
\begin{array}{cccccccc}
1 & 4 & -18 & -6 & 91 & -60 & 9 \\
4 & -14 & -20 & 71 & 1 & & \\
\hline
4 & -14 & -20 & 71 & 11 & 10 & \\
\hline
3 & 4 & -6 & -24 & 19 & -3 & 0 & \\
12 & 18 & -18 & 3 & & & \\
\hline
4 & 6 & -6 & 1 & 0 & & \Rightarrow x = 3 \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x - 3) \left( 4x^4 - 6x^3 - 24x^2 + 19x - 3 \right) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{cccc}
3 & 4 & -6 & -24 \\
12 & 18 & -18 & 3 \\
\hline
4 & 6 & -6 & 1 \\
\Rightarrow x = 3 \text{ is a zero again.}
\end{array}
\]

So \( P(x) = (x - 3)^2 \left( 4x^3 + 6x^2 - 6x + 1 \right) \). Continuing by factoring the quotient, we have:

\[
\begin{array}{cccc}
1 & 4 & 6 & -6 & 1 \\
12 & 54 & 144 & & \\
\hline
4 & 18 & 48 & 1445 & \Rightarrow x = 3 \text{ is an upper bound.}
\end{array}
\]

So \( P(x) = (x - 3)^2 \left( x - \frac{1}{2} \right) \left( 4x^2 + 8x - 2 \right) = 2 \left( x - 3 \right)^2 \left( x - \frac{1}{2} \right) \left( 2x^2 + 4x - 1 \right) \). Using the quadratic formula on the second factor, we have \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4 \cdot 2 \cdot 2}}{4} = -1 \pm \frac{\sqrt{6}}{2} \). Therefore, the zeros are \( x = \frac{1}{2}, 3, -1 \pm \frac{\sqrt{6}}{2} \).

51. (a) \( P(x) = x^3 - 3x^2 - 4x + 12 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \).

\[
\begin{array}{cccc}
1 & 1 & -3 & -4 \\
1 & 12 & -6 & & \\
\hline
1 & -2 & -6 & \\
2 & 1 & -3 & -4 \\
2 & 12 & -12 & & \\
\hline
1 & -1 & -6 & 0 & \Rightarrow x = 2 \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x - 2) \left( x^2 - x - 6 \right) = (x - 2)(x + 2)(x - 3) \). The real zeros of \( P \) are \( -2, 2, 3 \).

52. (a) \( P(x) = -x^3 - 2x^2 + 5x + 6 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 6 \).

\[
\begin{array}{cccc}
1 & -1 & -2 & 5 \\
1 & -1 & -3 & 2 \\
\hline
-1 & -3 & 2 & 8 \\
2 & -1 & -2 & 5 \\
2 & -2 & -8 & -6 \\
\hline
-1 & -4 & -3 & 0 & \Rightarrow x = 2 \text{ is a zero.}
\end{array}
\]

So \( P(x) = (x - 2) \left( -x^2 - 4x - 3 \right) = -(x - 2)(x^2 + 4x + 3) = -(x - 2)(x + 1)(x + 3) \). The real zeros of \( P \) are \( 2, -1, -3 \).
1. (a) \( P(x) = 2x^3 - 7x^2 + 4x + 4 \) has possible rational zeros \( \pm 1, \pm 2, \pm 4, \pm \frac{1}{2} \).

\[
\begin{array}{c|cccc}
1 & 2 & -7 & 4 & 4 \\
2 & 2 & -5 & -1 & 4 \\
\hline
2 & 2 & -5 & -1 & -3 \\
\end{array}
\] 
\( \Rightarrow x = 2 \) is a zero.

So \( P(x) = (x - 2) \left( 2x^2 - 3x - 2 \right) \). Continuing:

\[
\begin{array}{c|cccc}
2 & 2 & -3 & -2 \\
4 & 2 & 0 \\
\hline
2 & 1 & 0 \\
\end{array}
\] 
\( \Rightarrow x = 2 \) is a zero again.

Thus \( P(x) = (x - 2)^2 (2x + 1) \). The real zeros of \( P \) are \( 2 \) and \( -\frac{1}{2} \).

4. (a) \( P(x) = 3x^3 + 17x^2 + 21x - 9 \) has possible rational zeros \( \pm 1, \pm 3, \pm 9, \pm \frac{i}{3}, \pm \frac{1}{2} \).

\[
\begin{array}{c|cccc}
1 & 3 & 17 & 21 & -9 \\
3 & 20 & 41 & 32 \\
\hline
3 & 20 & 41 & 32 \\
\end{array}
\] 
\( \Rightarrow x = 1 \) is an upper bound.

\[
\begin{array}{c|cccc}
\frac{1}{3} & 3 & 17 & 21 & -9 \\
\frac{1}{3} & 1 & 6 & 9 \\
\hline
3 & 18 & 27 & 0 \\
\end{array}
\] 
\( \Rightarrow x = \frac{1}{3} \) is a zero.

So \( P(x) = (x - \frac{1}{3}) (3x^2 + 18x + 27) = 3 \left( x - \frac{1}{3} \right) \left( x^2 + 6x + 9 \right) = 3 \left( x - \frac{1}{3} \right) \left( x + 3 \right)^2 \). The real zeros of \( P \) are \(-3, \frac{1}{3}\).

55. (a) \( P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8 \) has possible rational zeros \( \pm 1, \pm 2, \pm 4, \pm 8 \).

\[
\begin{array}{c|cccc}
1 & 1 & -5 & 6 & 4 & -8 \\
1 & -4 & 2 & 6 \\
\hline
1 & -4 & 2 & 6 \\
\end{array}
\] 

\[
\begin{array}{c|cccc}
2 & 1 & -5 & 6 & 4 & -8 \\
2 & -6 & 0 & 8 \\
\hline
2 & -6 & 0 & 8 \\
\end{array}
\] 
\( \Rightarrow x = 2 \) is a zero.

So \( P(x) = (x - 2) \left( x^3 - 3x^2 + 4 \right) \) and the possible rational zeros are restricted to \(-1, \pm 2, \pm 4 \).

\[
\begin{array}{c|cccc}
2 & 1 & -3 & 0 & 4 \\
2 & -2 & -4 \\
\hline
2 & -2 & -4 \\
\end{array}
\] 
\( \Rightarrow x = 2 \) is a zero again.

\( P(x) = (x - 2)^2 \left( x^2 - x - 2 \right) = (x - 2)^2 (x - 2) (x + 1) = (x - 2)^3 (x + 1) \). So the real zeros of \( P \) are \(-1\) and \( 2 \).
56. (a) \( P(x) = -x^4 + 10x^2 + 8x - 8 \) has possible rational zeros ±1, ±2, ±4, ±8.

\[
\begin{array}{c|ccccc}
1 & -1 & 0 & 10 & 8 & -8 \\
-1 & 1 & 9 & 17 & \\
1 & 1 & 9 & 17 & 9 \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
4 & -1 & 0 & 10 & 8 & -8 \\
-4 & -16 & -24 & -64 & \\
-1 & -4 & -6 & -16 & -72 \\
\end{array}
\]

So \( P(x) = (x + 2)(-x^3 + 2x^2 + 6x - 4) \). Continuing, we have:

\[
\begin{array}{c|ccccc}
-2 & -1 & 2 & 6 & -4 \\
2 & 2 & -8 & 4 & \\
1 & 4 & -2 & 0 \\
\end{array}
\]

\( x = -2 \) is a zero again.

\( P(x) = (x + 2)^2(-x^2 + 4x - 2) \). Using the quadratic formula on the second factor, we have

\[
x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(-2)}}{2(-1)} = \frac{-4 \pm \sqrt{2}}{-2} = 2 \pm \sqrt{2}.
\]

The real zeros of \( P \) are \(-2, 2 \pm \sqrt{2}\).

57. (a) \( P(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4 \) has possible rational zeros ±1, ±2, ±4.

\[
\begin{array}{c|ccccc}
1 & 1 & -1 & -5 & 1 & 8 & 4 \\
1 & 0 & -5 & -4 & 4 & \\
1 & 0 & -5 & -4 & 4 & 8 \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
2 & 1 & -1 & -5 & 1 & 8 & 4 \\
2 & 2 & -6 & -10 & -4 & \\
1 & 1 & -3 & -5 & -2 & 0 \\
\end{array}
\]

\( x = 2 \) is a zero.

So \( P(x) = (x - 2)(x^4 + x^3 - 3x^2 - 5x - 2) \), and the possible rational zeros are restricted to -1, ±2.

\[
\begin{array}{c|ccccc}
2 & 1 & 1 & -3 & -5 & -2 \\
2 & 6 & 6 & 2 & \\
1 & 3 & 3 & 1 & 0 \\
\end{array}
\]

\( x = 2 \) is a zero again.

So \( P(x) = (x - 2)^2(x^3 + 3x^2 + 3x + 1) \), and the possible rational zeros are restricted to -1.

\[
\begin{array}{c|ccccc}
-1 & 1 & 3 & 3 & 1 \\
-1 & -2 & -1 & \\
1 & 2 & 1 & 0 \\
\end{array}
\]

\( x = -1 \) is a zero.

So \( P(x) = (x - 2)^2(x + 1)(x^2 + 2x + 1) = (x - 2)^2(x + 1)^3 \), and the real zeros of \( P \) are -1 and 2.
8. (a) \( P(x) = x^5 - x^4 - 6x^3 + 14x^2 - 11x + 3 \) has possible rational zeros \( \pm 1, \pm 3 \).

\[
\begin{array}{c|cccccc}
1 & 1 & -1 & -6 & 14 & -11 & 3 \\
\hline
1 & 0 & -6 & 8 & -3 & \\
1 & 0 & -6 & 8 & -3 & 0 \\
\end{array}
\Rightarrow x = 1 \text{ is a zero.}
\]

So \( P(x) = (x-1)(x^4 - 6x^2 + 8x - 3) \):

\[
\begin{array}{c|cccc}
1 & 1 & -6 & 8 & -3 \\
\hline
1 & 1 & -5 & 3 & \\
1 & 1 & -5 & 3 & 0 \\
\end{array}
\Rightarrow x = 1 \text{ is a zero again.}
\]

So \( P(x) = (x-1)^2(x^3 + x^2 - 5x + 3) \):

\[
\begin{array}{c|cccc}
1 & 1 & -5 & 3 \\
\hline
1 & 2 & -3 & \\
1 & 2 & -3 & 0 \\
\end{array}
\Rightarrow x = 1 \text{ is a zero again.}
\]

So \( P(x) = (x-1)^3(x^2 + 2x - 3) = (x-1)^4(x+3) \), and the real zeros of \( P \) are 1 and \(-3\).

59. \( P(x) = x^3 - x^2 - x - 3 \). Since \( P(x) \) has 1 variation in sign, \( P \) has 1 positive real zero. Since \( P(-x) = -x^3 - x^2 + x - 3 \) has 2 variations in sign, \( P \) has 0 or 2 negative real zeros. Thus, \( P \) has 1 or 3 real zeros.

60. \( P(x) = 2x^3 - x^2 + 4x - 7 \). Since \( P(x) \) has 3 variations in signs, \( P \) has 3 or 1 positive real zeros. Since \( P(-x) = -2x^3 - x^2 - 4x - 7 \) has no variation in sign, there is no negative real zero. Thus, \( P \) has 1 or 3 real zeros.

61. \( P(x) = 2x^6 + 5x^4 - x^3 - 5x - 1 \). Since \( P(x) \) has 1 variation in sign, \( P \) has 1 positive real zero. Since \( P(-x) = 2x^6 + 5x^4 + x^3 + 5x - 1 \) has 1 variation in sign, \( P \) has 1 negative real zero. Therefore, \( P \) has 2 real zeros.

62. \( P(x) = x^4 + x^3 + x^2 + x + 12 \). Since \( P(x) \) has no variations in sign, \( P \) has no positive real zeros. Since \( P(-x) = x^4 - x^3 + x^2 - x + 12 \) has 4 variations in sign, \( P \) has 4, 2, or 0 negative real zeros. Therefore, \( P(x) \) has 0, 2, or 4 real zeros.

63. \( P(x) = x^5 + 4x^3 - x^2 + 6x \). Since \( P(x) \) has 2 variations in sign, \( P \) has 2 or 0 positive real zeros. Since \( P(-x) = -x^5 - 4x^3 - x^2 - 6x \) has no variation in sign, \( P \) has no negative real zero. Therefore, \( P \) has a total of 1 or 3 real zeros (since \( x = 0 \) is a zero, but is neither positive nor negative).

64. \( P(x) = x^8 - x^5 + x^4 - x^3 + x^2 - x + 1 \). Since \( P(x) \) has 6 variations in sign, the polynomial has 6, 4, 2, or 0 positive real zeros. Since \( P(-x) \) has no variation in sign, the polynomial has no negative real zeros. Therefore, \( P \) has 6, 4, 2, or 0 real zeros.

65. \( P(x) = 2x^3 + 5x^2 + x - 2; a = -3, b = 1 \)

\[
\begin{array}{c|cccc}
-3 & 2 & 5 & 1 & -2 \\
\hline
2 & 3 & -12 & \\
2 & 3 & -12 & 0 \\
\end{array}
\Rightarrow \text{lower bound.}
\]

\[
\begin{array}{c|cccc}
1 & 2 & 5 & 1 & -2 \\
\hline
2 & 7 & 8 & \\
2 & 7 & 8 & 6 \\
\end{array}
\Rightarrow \text{upper bound.}
\]

Therefore \( a = -3 \) and \( b = 1 \) are lower and upper bounds.
66. \( P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8; \ a = -3, \ b = 5 \)

\[
\begin{array}{rrrrr}
  & 1 & -2 & -9 & 2 & 8 \\
\hline
-3 &  & 15 & -18 & 48 \\
\hline
  & 1 & -5 & 6 & -16 & 56 \\
\end{array}
\]

Alternating signs \( \Rightarrow \) lower bound.

\[
\begin{array}{rrrrr}
  & 1 & -2 & -9 & 2 & 8 \\
\hline
5 &  & 15 & 30 & 160 \\
\hline
  & 1 & 3 & 6 & 32 & 168 \\
\end{array}
\]

All nonnegative \( \Rightarrow \) upper bound.

Therefore \( a = -3 \) and \( b = 5 \) are lower and upper bounds.

67. \( P(x) = 8x^3 + 10x^2 - 39x + 9; \ a = -3, \ b = 2 \)

\[
\begin{array}{rrrrr}
  & 8 & 10 & -39 & 9 \\
\hline
-3 &  & 24 & 42 & -9 \\
\hline
  & 8 & -14 & 3 & 0 \\
\end{array}
\]

Alternating signs \( \Rightarrow \) lower bound.

\[
\begin{array}{rrrrr}
  & 8 & 10 & -39 & 9 \\
\hline
2 &  & 16 & 52 & 26 \\
\hline
  & 8 & 26 & 13 & 35 \\
\end{array}
\]

All nonnegative \( \Rightarrow \) upper bound.

Therefore \( a = -3 \) and \( b = 2 \) are lower and upper bounds. Note that \( x = -3 \) is also a zero.

68. \( P(x) = 3x^4 - 17x^3 + 24x^2 - 9x + 1; \ a = 0, \ b = 6 \)

\[
\begin{array}{rrrrr}
  & 3 & -17 & 24 & -9 & 1 \\
\hline
0 &  & 0 & 0 & 0 & 0 \\
\hline
  & 3 & -17 & 24 & -9 & 1 \\
\end{array}
\]

Alternating signs \( \Rightarrow \) lower bound.

\[
\begin{array}{rrrrr}
  & 3 & -17 & 24 & -9 & 1 \\
\hline
6 &  & 18 & 6 & 180 & 1026 \\
\hline
  & 3 & 1 & 30 & 171 & 1027 \\
\end{array}
\]

All nonnegative \( \Rightarrow \) upper bound.

Therefore \( a = 0, \ b = 6 \) are lower and upper bounds, respectively. Note, since \( P(x) \) alternates in sign, by Descartes' Rule of Signs, \( 0 \) is automatically a lower bound.

69. \( P(x) = x^3 - 3x^2 + 4 \) and use the Upper and Lower Bounds Theorem:

\[
\begin{array}{rrrrr}
  & 1 & -3 & 0 & 4 \\
\hline
-1 &  & 4 & -4 \\
\hline
  & 1 & -4 & 4 & 0 \\
\end{array}
\]

Alternating signs \( \Rightarrow \) lower bound.

\[
\begin{array}{rrrrr}
  & 1 & -3 & 0 & 4 \\
\hline
3 &  & 3 & 0 & 0 \\
\hline
  & 1 & 0 & 0 & 4 \\
\end{array}
\]

All nonnegative \( \Rightarrow \) upper bound.

Therefore \(-1\) is a lower bound (and a zero) and \(3\) is an upper bound. (There are many possible solutions.)
70. $P(x) = 2x^3 - 3x^2 - 8x + 12$ and using the Upper and Lower Bounds Theorem:

\[
\begin{array}{cccc}
-2 & 2 & -3 & -8 & 12 \\
& -4 & 14 & -12 \\
2 & -7 & 6 & 0 & \text{Alternating signs } \Rightarrow \ x = -2 \text{ is a lower bound (and a zero).} \\
3 & 2 & -3 & -8 & 12 \\
& 6 & 9 & 3 \\
2 & 3 & 1 & 15 & \text{All nonnegative } \Rightarrow \ x = 3 \text{ is an upper bound.}
\end{array}
\]

(There are many possible solutions.)

71. $P(x) = x^4 - 2x^3 + x^2 - 9x + 2$

\[
\begin{array}{cccc}
1 & 1 & -2 & 1 & -9 & 2 \\
& 1 & -1 & 0 & -9 \\
1 & -1 & 0 & -9 & -7 \\
3 & 1 & -2 & 1 & -9 & 2 \\
& 3 & 3 & 12 & 9 \\
1 & 1 & 4 & 3 & 11 & \text{all positive } \Rightarrow \text{ upper bound.} \\
-1 & 1 & -2 & 1 & -9 & 2 \\
& -1 & 3 & -4 & 13 \\
1 & -3 & 4 & -13 & 15 & \text{alternating signs } \Rightarrow \text{ lower bound.}
\end{array}
\]

Therefore $-1$ is a lower bound and $3$ is an upper bound. (There are many possible solutions.)

72. Set $P(x) = x^5 - x^4 + 1$.

\[
\begin{array}{cccc}
1 & 1 & -1 & 0 & 0 & 0 & 1 \\
& 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & \text{All nonnegative } \Rightarrow \ x = 1 \text{ is an upper bound.} \\
-1 & 1 & -1 & 0 & 0 & 0 & 1 \\
& -1 & 2 & -2 & 2 & -2 \\
1 & -2 & 2 & -2 & 2 & -1 & \text{Alternating signs } \Rightarrow \ x = -1 \text{ is a lower bound.}
\end{array}
\]

(There are many possible solutions.)

73. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$.

\[
\begin{array}{cccc}
1 & 2 & 3 & -4 & -3 & 2 \\
& 2 & 5 & 1 & -2 \\
2 & 5 & 1 & -2 & 0 & \Rightarrow x = 1 \text{ is a zero.}
\end{array}
\]

$P(x) = (x - 1)(2x^3 + 5x^2 + x - 2)$

\[
\begin{array}{cccc}
-1 & 2 & 5 & 1 & -2 \\
& -2 & -3 & 2 \\
2 & 3 & -2 & 0 & \Rightarrow x = -1 \text{ is a zero.}
\end{array}
\]

$P(x) = (x - 1)(x + 1)(2x^2 + 3x - 2) = (x - 1)(x + 1)(2x - 1)(x + 2)$. Therefore, the zeros are $x = -2, \frac{1}{2}, \pm 1$. 

74. \( P(x) = 2x^4 + 15x^3 + 31x^2 + 20x + 4 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 4, \pm \frac{1}{2} \). Since all of the coefficients are positive, there are no positive zeros. Since \( P(-x) = 2x^4 - 15x^3 + 31x^2 - 20x + 4 \) has 4 variations in sign, there are 0, 2, or 4 negative real zeros.

\[
\begin{array}{rrrrr}
-1 & 2 & 15 & 31 & 20 & 4 \\
-2 & -13 & -18 & -2 & \\
2 & 13 & 18 & 2 & 2 \\
\end{array}
\quad
\begin{array}{rrrrr}
-2 & 2 & 15 & 31 & 20 & 4 \\
-4 & -22 & -18 & -4 & \\
2 & 11 & 9 & 2 & 0
\end{array}
\]

\( P(x) = (x+2) (2x^3 + 11x^2 + 9x + 2) \):

\[
\begin{array}{rrrrr}
-2 & 2 & 11 & 9 & 2 \\
-4 & -14 & 10 & \\
2 & 7 & -5 & 12 \\
\end{array}
\quad
\begin{array}{rrrrr}
-4 & 2 & 11 & 9 & 2 \\
-8 & -12 & 12 & \\
2 & 3 & -3 & 14 \\
\end{array}
\quad
\begin{array}{rrrrr}
-2 & 2 & 11 & 9 & 2 \\
-1 & -5 & -2 & \\
2 & 10 & 4 & 0
\end{array}
\Rightarrow x = -2 \text{ is a zero.}

\[
\begin{array}{rrrrr}
2 & 12 & 11 & 9 & 2 \\
-4 & -14 & 10 & \\
2 & 7 & -5 & 12 \\
\end{array}
\Rightarrow x = -\frac{1}{2} \text{ is a zero.}
\]

\[
P(x) = (x+2) (2x+1) (x^2 + 5x + 2). \text{ Now if } x^2 + 5x + 2 = 0, \text{ then } x = -5 \pm \sqrt{25 - 4(1)(2)} = -5 \pm \sqrt{17}. \text{ Thus, the zeros are } -2, -\frac{1}{2}, \text{ and } -5 \pm \sqrt{17}.
\]

75. Method 1: \( P(x) = 4x^4 - 21x^2 + 5 \) has 2 variations in sign, so by Descartes’ rule of signs there are either 2 or 0 positive zeros. If we replace \( x \) with \( -x \), the function does not change, so there are either 2 or 0 negative zeros. Possible rational zeros are \( \pm 1, \pm \frac{1}{2}, \pm \frac{5}{2}, \pm 5 \). By inspection, \( \pm 1 \) and \( \pm 5 \) are not zeros, so we must look for non-integer solutions:

\[
\begin{array}{rrrrr}
\frac{1}{2} & 4 & 0 & -21 & 0 & 5 \\
2 & 1 & -10 & -5 & \\
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

\[
P(x) = (x - \frac{1}{2}) (4x^3 + 2x^2 - 20x - 10), \text{ continuing with the quotient, we have:}
\]

\[
\begin{array}{rrrrr}
-\frac{1}{2} & 4 & 2 & -20 & -10 \\
-2 & 0 & 10 & \\
\end{array}
\Rightarrow x = -\frac{1}{2} \text{ is a zero.}
\]

\[
P(x) = (x - \frac{1}{2}) \left( x + \frac{1}{2} \right) (4x^2 - 20) = 0. \text{ If } 4x^2 - 20 = 0, \text{ then } x = \pm \sqrt{5}. \text{ Thus the zeros are } x = \pm \frac{1}{2}, \pm \sqrt{5}.
\]

Method 2: Substituting \( u = x^2 \), the equation becomes \( 4u^2 - 21u + 5 = 0 \), which factors:

\[
4u^2 - 21u + 5 = (4u - 1)(u - 5) = (4x^2 - 1)(x^2 - 5). \text{ Then either we have } x^2 = 5, \text{ so that } x = \pm \sqrt{5}, \text{ or we have}
\]

\[
x^2 = \frac{1}{4}, \text{ so that } x = \pm \sqrt{\frac{1}{4}} = \pm \frac{1}{2}. \text{ Thus the zeros are } x = \pm \frac{1}{2}, \pm \sqrt{5}.
\]

76. \( P(x) = 6x^4 - 7x^3 - 8x^2 + 5x = x(6x^3 - 7x^2 - 8x + 5) \). So \( x = 0 \) is a zero. Continuing with the quotient, \( Q(x) = 6x^3 - 7x^2 - 8x + 5 \). The possible rational zeros are \( \pm 1, \pm 5, \pm \frac{1}{3}, \pm \frac{5}{3}, \pm \frac{1}{6}, \pm \frac{5}{6} \). Since \( Q(x) \) has 2 variations in sign, there are 0 or 2 positive real zeros. Since \( Q(-x) = 6x^4 + 7x^3 - 8x^2 - 5x \) has 1 variation in sign, there is 1 negative real zero.

\[
\begin{array}{rrrrr}
1 & 6 & -7 & -8 & 5 \\
6 & -1 & -9 & \\
6 & -1 & -9 & -4 \\
\end{array}
\quad
\begin{array}{rrrrr}
5 & 6 & -7 & -8 & 5 \\
30 & 115 & 535 & \\
6 & 23 & 107 & 540
\end{array}
\text{ All positive } \Rightarrow \text{ upper bound.}
\]

\[
\begin{array}{rrrrr}
\frac{1}{2} & 6 & -7 & -8 & 5 \\
3 & -2 & -5 & \\
6 & -4 & -10 & 0
\end{array}
\Rightarrow x = \frac{1}{2} \text{ is a zero.}
\]

\[
P(x) = x (2x - 1) (3x^2 - 2x - 5) = x (2x - 1) (3x - 5) (x + 1). \text{ Therefore, the zeros are } 0, -1, \frac{1}{2} \text{ and } \frac{5}{3}.
\]
77. \( P(x) = x^5 - 7x^4 + 9x^3 + 23x^2 - 50x + 24 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \). \( P(x) \) has 4 variations in sign and hence 0, 2, or 4 positive real zeros. \( P(-x) = -x^5 - 7x^4 - 9x^3 + 23x^2 + 50x + 24 \) has 1 variation in sign, and hence 1 negative real zero.

\[
\begin{array}{c|cccccc}
1 & 1 & -7 & 9 & 23 & -50 & 24 \\
1 & -6 & 3 & 26 & -24 & \\
\hline
1 & -6 & 3 & 26 & -24 & 0 \\
\Rightarrow x = 1 \text{ is a zero.}
\end{array}
\]

\( P(x) = (x - 1)(x^4 - 6x^3 + 3x^2 + 26x - 24) \); continuing with the quotient, we try 1 again.

\[
\begin{array}{c|cccccc}
1 & 1 & -6 & 3 & 26 & -24 \\
1 & -5 & -2 & 24 & \\
\hline
1 & -5 & -2 & 24 & 0 \\
\Rightarrow x = 1 \text{ is a zero again.}
\end{array}
\]

\( P(x) = (x - 1)^2(x^3 - 5x^2 - 2x + 24) \); continuing with the quotient, we start by trying 1 again.

\[
\begin{array}{c|cccccc}
1 & 1 & -5 & -2 & 24 & 2 & 1 & -5 & -2 & 24 & 3 & 1 & -5 & -2 & 24 \\
1 & -4 & -6 & 18 & & 1 & -3 & -8 & 8 & 1 & -2 & -8 & 0 & \\
\hline
1 & -4 & -6 & 18 & 0 & -57 & 92 & 8 & -30 & 38 & -19 & 3 & 0 & \\
\Rightarrow x = 3 \text{ is a zero.}
\end{array}
\]

\( P(x) = (x - 1)^2(x - 3)(x^2 - 2x - 8) = (x - 1)^2(x - 3)(x - 4)(x + 2) \). Therefore, the zeros are \( x = -2, 1, 3, 4 \).

78. \( P(x) = 8x^5 - 14x^4 - 22x^3 + 57x^2 - 35x + 6 \). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm \frac{1}{3}, \pm \frac{1}{6} \). Since \( P(x) \) has 4 variations in sign, there are 0, 2, or 4 positive real zeros. Since \( P(-x) = -8x^5 - 14x^4 + 22x^3 + 57x^2 + 35x + 6 \) has 1 variation in sign, there is 1 negative real zero.

\[
\begin{array}{c|cccccc}
-1 & 8 & -14 & -22 & 57 & -35 & 6 \\
-8 & 22 & 0 & -57 & 92 & \\
\hline
8 & -22 & 0 & 57 & -92 & 98 & 8 & -30 & 38 & -19 & 3 & 0 & \\
\Rightarrow x = -2 \text{ is a zero.}
\end{array}
\]

\( P(x) = (x + 2)(8x^4 - 30x^3 + 38x^2 - 19x + 3) \). All the other real zeros are positive.

\[
\begin{array}{c|cccccc}
1 & 8 & -30 & 38 & -19 & 3 \\
8 & -22 & 0 & 16 & -3 & \\
\hline
8 & -22 & 16 & -3 & 0 \\
\Rightarrow x = 1 \text{ is a zero.}
\end{array}
\]

\( P(x) = (x + 2)(x - 1)(8x^3 - 22x^2 + 16x - 3) \).

\[
\begin{array}{c|cccccc}
1 & 8 & -22 & 16 & -3 \\
8 & -14 & 2 & \\
\hline
8 & -14 & 2 & -1 & \\
\Rightarrow x = \frac{1}{2} \text{ and } 1. \text{ We try } \frac{3}{4}: \\
\frac{3}{4} & 8 & -22 & 16 & -3 & \\
6 & -12 & 3 & \\
\Rightarrow x = \frac{3}{4} \text{ is a zero.}
\end{array}
\]

\( P(x) = (x + 2)(x - 1)(4x - 3)(2x^2 - 4x + 1) \). Now, \( 2x^2 - 4x + 1 = 0 \) when \( x = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2(2)} = \frac{2 \pm \sqrt{2}}{2} \). Thus, the zeros are \( 1, \frac{3}{4}, -2, \text{ and } \frac{2 + \sqrt{2}}{2} \).
79. \( P(x) = x^3 - x - 2 \). The only possible rational zeros of \( P(x) \) are \( \pm 1 \) and \( \pm 2 \).

\[
\begin{array}{c|cccc}
1 & 1 & 0 & -1 & -2 \\
\hline
1 & 1 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

Since the row that contains \(-1\) alternates between nonnegative and nonpositive, \(-1\) is a lower bound and there is no need to try \(-2\). Therefore, \( P(x) \) does not have any rational zeros.

80. \( P(x) = 2x^4 - x^3 + x + 2 \). The only possible rational zeros of \( P(x) \) are \( \pm 1, \pm 2, \pm \frac{1}{2} \).

\[
\begin{array}{c|cccc}
\frac{1}{2} & 2 & -1 & 0 & 1 & 2 \\
\hline
1 & 0 & 0 & \frac{1}{2} \\
2 & 0 & 0 & 1 & \frac{5}{2} \\
\end{array}
\]

All nonnegative \( \Rightarrow x = \frac{1}{2} \) is an upper bound.

\[
\begin{array}{c|cccc}
-1 & 2 & -1 & 0 & 1 & 2 \\
\hline
-2 & 3 & -3 & 2 \\
-1 & -3 & 3 & -2 & 4 \\
\end{array}
\]

Alternating signs \( \Rightarrow x = -1 \) is a lower bound.

\[
\begin{array}{c|cccc}
-\frac{1}{2} & 2 & -1 & 0 & 1 & 2 \\
\hline
-1 & 1 & -\frac{1}{2} & -\frac{1}{4} \\
2 & -2 & 1 & \frac{1}{2} & \frac{7}{4} \\
\end{array}
\]

Therefore, there is no rational zero.

81. \( P(x) = 3x^3 - x^2 - 6x + 12 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3} \).

\[
\begin{array}{c|cccc}
3 & -1 & -6 & 12 \\
\hline
1 & 3 & 2 & -4 & 8 \\
2 & 3 & 5 & 4 & 20 \\
-1 & 3 & -4 & -2 & 14 \\
-2 & 3 & -7 & 8 & -4 \\
\end{array}
\]

All positive \( \Rightarrow x = 2 \) is an upper bound.

\[
\begin{array}{c|cccc}
3 & -1 & -6 & 12 \\
\hline
\frac{1}{3} & 3 & 0 & -6 & 10 \\
\frac{2}{3} & 3 & 1 & -\frac{16}{3} & \frac{76}{9} \\
\frac{4}{3} & 3 & 3 & -2 & \frac{28}{3} \\
-\frac{1}{3} & 3 & -2 & -\frac{16}{3} & \frac{124}{9} \\
-\frac{2}{3} & 3 & -3 & -4 & \frac{44}{3} \\
-\frac{4}{3} & 3 & -5 & \frac{2}{3} & \frac{100}{9} \\
\end{array}
\]

Therefore, there is no rational zero.

82. \( P(x) = x^{50} - 5x^{25} + x^2 - 1 \). The only possible rational zeros of \( P(x) \) are \( \pm 1 \). Since \( P(1) = (1)^{50} - 5(1)^{25} + (1)^2 - 1 = -4 \) and \( P(-1) = (-1)^{50} - 5(-1)^{25} + (-1)^2 - 1 = 6 \), \( P(x) \) does not have a rational zero.

83. \( P(x) = x^3 - 3x^2 - 4x + 12 \), \([-4, 4]\) by \([-15, 15]\). The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \). By observing the graph of \( P \), the rational zeros are \( x = -2, 2, 3 \).

84. \( P(x) = x^4 - 5x^2 + 4 \), \([-4, 4]\) by \([-30, 30]\). The possible rational solutions are \( \pm 1, \pm 2, \pm 4 \).

By observing the graph of the equation, the solutions of the given equation are \( x = \pm 1, \pm 2 \).
15. \( P(x) = 2x^4 - 5x^3 - 14x^2 + 5x + 12 \), \([-2, 5]\) by \([-40, 40]\). The possible rational zeros are 
\[\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}.\] By observing the graph of \( P \), the zeros are \( x = -\frac{3}{2}, -1, 1, 4.\)

86. \( P(x) = 3x^3 + 8x^2 + 5x + 2 \), \([-3, 3]\) by \([-10, 10]\). The possible rational solutions are \( \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}.\) By observing the graph of the equation, the only real solution of the given equation is \( x = -2.\)

37. \( x^4 - x - 4 = 0 \). Possible rational solutions are \( \pm 1, \pm 2, \pm 4.\)

\[
\begin{array}{cccc}
1 & 0 & 0 & -1 & -4 \\
1 & 1 & 1 & 0 & -4 \\
-1 & 1 & 1 & 2 \\
1 & -1 & 1 & -2 & -2 \\
\end{array}
\]

Therefore, we graph the function \( P(x) = x^4 - x - 4 \) in the viewing rectangle \([-2, 2]\) by \([-5, 20]\) and see there are two solutions. In the viewing rectangle \([-1.3, -1.25]\) by \([-0.1, 0.1]\), we find the solution \( x \approx -1.28.\) In the viewing rectangle \([1.5, 1.6]\) by \([-0.1, 0.1]\), we find the solution \( x \approx 1.53.\) Thus the solutions are \( x \approx -1.28, 1.53.\)

88. \( 2x^3 - 8x^2 + 9x - 9 = 0 \). Possible rational solutions are \( \pm 1, \pm 2, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}.\)

\[
\begin{array}{cccc}
1 & 2 & -8 & 9 & -9 \\
2 & -6 & 3 & -6 \\
2 & -6 & 3 & -6 \\
\end{array}
\]

We graph \( P(x) = 2x^3 - 8x^2 + 9x - 9 \) in the viewing rectangle \([-4, 6]\) by \([-40, 40]\). It appears that the equation has no other real solution. We can factor 
\( 2x^3 - 8x^2 + 9x - 9 = (x - 3) (2x^2 - 2x + 3).\) Since the quotient is a quadratic expression, we can use the quadratic formula to locate the other possible solutions:

\[ x = \frac{2 \pm \sqrt{2^2 - 4(2)(3)}}{2(2)} \], which are not real. So the only solution is \( x = 3.\)
89. $4.00x^4 + 4.00x^3 - 10.96x^2 - 5.88x + 9.09 = 0$.

\[
\begin{array}{cccccc}
1 & 4 & 4 & -10.96 & -5.88 & 9.09 \\
4 & 8 & -2.96 & -8.84 & 0.25 \\
4 & 8 & -2.96 & -8.84 & 0.25 \\
\end{array}
\begin{array}{cccccc}
2 & 4 & 4 & -10.96 & -5.88 & 9.09 \\
4 & 8 & 24 & 26.08 & 40.40 \\
4 & 12 & 13.04 & 20.2 & 49.49 \\
\end{array}
\Rightarrow x = 2 \text{ is an upper bound.}

\begin{array}{cccccc}
-2 & 4 & -10.96 & -5.88 & 9.09 \\
-8 & 8 & 5.92 & -0.08 \\
4 & -4 & -2.96 & 0.04 \\
\end{array}
\begin{array}{cccccc}
-3 & 4 & -10.96 & -5.88 & 9.09 \\
-8 & 8 & -2.96 & -8.84 \\
4 & -4 & -2.96 & 0.04 \\
\end{array}
\Rightarrow x = -3 \text{ is a lower bound.}

Therefore, we graph the function $P(x) = 4.00x^4 + 4.00x^3 - 10.96x^2 - 5.88x + 9.09$ in the viewing rectangle $[-3, 2]$ by $[-10, 40]$. There appear to be two solutions. In the viewing rectangle $[-1.6, -1.4]$ by $[0, 1]$, we find the solution $x \approx -1.50$. In the viewing rectangle $[0.8, 1.2]$ by $[0, 1]$, we see that the graph comes close but does not go through the $x$-axis. Thus there is no solution here. Therefore, the only solution is $x \approx -1.50$.

90. $x^5 + 2x^4 + 0.96x^3 + 5x^2 + 10x + 4.8 = 0$. Since all the coefficients are positive, there is no positive solution. So $x = 0$ is an upper bound.

\[
\begin{array}{ccccccc}
2 & 1 & 2 & 0.96 & 5 & 10 & 4.8 \\
-2 & 0 & -1.92 & -6.16 & -7.68 \\
1 & 0 & 0.96 & 3.08 & 3.84 & -2.88 \\
\end{array}
\begin{array}{ccccccc}
3 & 1 & 2 & 0.96 & 5 & 10 & 4.8 \\
-3 & 3 & -11.88 & 20.64 & -91.92 \\
1 & -1 & 3.96 & -6.88 & 30.64 & -87.12 \\
\end{array}
\Rightarrow x = -3 \text{ is a lower bound.}

Therefore, we graph $P(x) = x^5 + 2x^4 + 0.96x^3 + 5x^2 + 10x + 4.8$ in the viewing rectangle $[-3, 0]$ by $[-10, 5]$ and see that there are three possible solutions. In the viewing rectangle $[-1.75, -1.7]$ by $[-0.1, 0.1]$, we find the solution $x \approx -1.71$. In the viewing rectangle $[-1.25, -1.15]$ by $[-0.1, 0.1]$, we find the solution $x \approx -1.20$. In the viewing rectangle $[-0.85, -0.75]$ by $[-0.1, 0.1]$, we find the solution $x \approx -0.80$. So the solutions are $x \approx -1.71, -1.20, -0.80$.

91. (a) Since $z > b$, we have $z - b > 0$. Since all the coefficients of $Q(x)$ are nonnegative, and since $z > 0$, we have $Q(z) > 0$ (being a sum of positive terms). Thus, $P(z) = (z - b) \cdot Q(z) + r > 0$, since the sum of a positive number and a nonnegative number.

(b) In part (a), we showed that if $b$ satisfies the conditions of the first part of the Upper and Lower Bounds Theorem and $z > b$, then $P(z) > 0$. This means that no real zero of $P$ can be larger than $b$, so $b$ is an upper bound for the real zeros.

(c) Suppose $-b$ is a negative lower bound for the real zeros of $P(x)$. Then clearly $b$ is an upper bound for $P_1(x) = P(-x)$. Thus, as in Part (a), we can write $P_1(x) = (x - b) \cdot Q(x) + r$, where $r > 0$ and the coefficients of $Q$ are all nonnegative, and $P(x) = P_1(-x) = (-x + b) \cdot [-Q(-x)] + r$. Since the coefficients of $Q(x)$ are all nonnegative, the coefficients of $-Q(-x)$ will be alternately nonpositive and nonnegative, which proves the second part of the Upper and Lower Bounds Theorem.
12. \( P(x) = x^5 - x^4 - x^3 - 5x^2 - 12x - 6 \) has possible rational zeros \( \pm 1, \pm 2, \pm 3, \pm 6 \). Since \( P(x) \) has 1 variation in sign, there is 1 positive real zero. Since \( P(-x) = -x^5 - x^4 + x^3 - 5x^2 + 12x - 6 \) has 4 variations in sign, there are 0, 2, or 4 negative real zeros.

\[
\begin{array}{c|cccc}
1 & 1 & -1 & -1 & -5 \\
0 & 0 & -1 & -6 & -12 \\
1 & 1 & -1 & -6 & -18 \\
0 & 0 & -1 & -6 & -12 \\
1 & 1 & -1 & -6 & -18 \\
0 & 0 & -1 & -6 & -12 \\
\end{array}
\]

1 2 5 10 18 48 \( \Rightarrow \) 3 is an upper bound.

\( P(x) = (x+1)(x^4 - 2x^3 + x^2 - 6x - 6) \), continuing with the quotient we have

\[
\begin{array}{c|cccc}
1 & 1 & -1 & -6 & -6 \\
-1 & 0 & -3 & 4 & 10 \\
1 & 1 & -1 & -6 & -18 \\
0 & 0 & -1 & -6 & -12 \\
1 & 1 & -1 & -6 & -18 \\
0 & 0 & -1 & -6 & -12 \\
\end{array}
\]

Therefore, there is 1 rational zero, namely \(-1\). Since there are 1, 3 or 5 real zeros, and we found 1 rational zero, there must be 0, 2 or 4 irrational zeros. However, since 1 zero must be positive, there cannot be 0 irrational zeros. Therefore, there is exactly 1 rational zero and 2 or 4 irrational zeros.

93. Let \( r \) be the radius of the silo. The volume of the hemispherical roof is \( \frac{1}{2} \left( \frac{2}{3} \pi r^3 \right) = \frac{2}{3} \pi r^3 \). The volume of the cylindrical section is \( \pi r^2 (30) = 30 \pi r^2 \). Because the total volume of the silo is 15,000 ft\(^3\), we get the following equation:

\[
\frac{2}{3} \pi r^3 + 30 \pi r^2 = 15000 \quad \Rightarrow \quad \frac{2}{3} \pi r^3 + 30 \pi r^2 - 15000 = 0 \quad \Rightarrow \quad \pi r^3 + 45 \pi r^2 - 22500 = 0.
\]

Using a graphing device, we first graph the polynomial in the viewing rectangle \([0, 15]\) by \([-10000, 10000]\). The solution, \( r \approx 11.28 \) ft., is shown in the viewing rectangle \([11.2, 11.4]\) by \([-1, 1]\).

94. Given that \( x \) is the length of a side of the rectangle, we have that the length of the diagonal is \( x+10 \), and the length of the other side of the rectangle is \( \sqrt{(x+10)^2 - x^2} \). Hence \( x \sqrt{(x+10)^2 - x^2} = 5000 \quad \Rightarrow \quad x^2 (20x + 100) = 25,000,000 \quad \Rightarrow \quad 2x^3 + 10x^2 - 2,500,000 = 0 \quad \Rightarrow \quad x^2 + 5x^2 - 1,250,000 = 0 \). The first viewing rectangle, \([0, 120]\) by \([-100, 500]\), shows there is one solution. The second viewing rectangle, \([106, 106.1]\) by \([-0.1, 0.1]\), shows the solution is \( x = 106.08 \). Therefore, the dimensions of the rectangle are 47 ft by 106 ft.
95. \( h(t) = 11.60t - 12.41t^2 + 6.20t^3 - 1.58t^4 - 0.20t^5 - 0.01t^6 \)
is shown in the viewing rectangle 
[0, 10] by [0, 6].

(a) It started to snow again.
(b) No, \( h(t) \leq 4 \).
(c) The function \( h(t) \) is shown in the viewing rectangle [6, 6.5] by 
[0, 0.5]. The x-intercept of the function is a little less than 6.5, which
means that the snow melted just before midnight on Saturday night.

96. The volume of the box is 
\[ V = 1500 = x(20-2x)(40-2x) = 4x^3 - 120x^2 + 800x \]
\[ 4x^3 - 120x^2 + 800x - 1500 = 4(x^3 - 30x^2 + 200x - 375) = 0. \]
Clearly, we must have \( 20 - 2x > 0 \), and so
\( 0 < x < 10 \).

\[
\begin{array}{c|ccc}
5 & 1 & -30 & 200 & -375 \\
5 & 1 & -25 & 75 & 0 \\ 
\end{array}
\]
\( x^3 - 30x^2 + 200x - 375 = (x-5)(x^2 - 25x + 75) = 0. \)
Using the quadratic formula, we find the
other zeros: \( x = \frac{25 \pm \sqrt{625 - 4(1)(75)}}{2} = \frac{25 \pm \sqrt{125}}{2} = \frac{25 \pm 5\sqrt{13}}{2} \).
Since \( \frac{25 - 5\sqrt{13}}{2} > 10 \), the two answers are:
\( x = \text{height} = 5 \text{ cm}, \text{ width} = 20 - 2(5) = 10 \text{ cm}, \text{ and length} = 40 - 2(5) = 30 \text{ cm}; \) and \( x = \frac{25 - 5\sqrt{13}}{2} \),
width = \( 20 - (25 - 5\sqrt{13}) = 5\sqrt{13} - 5 \text{ cm}, \) and length = \( 40 - (25 - 5\sqrt{13}) = 15 + 5\sqrt{13} \text{ cm}. \)

97. Let \( r \) be the radius of the cone and cylinder and let \( h \) be the height of the cone.
Since the height and diameter are equal, we get \( h = 2r \). So the volume of the
cylinder is \( V_1 = \pi r^2 \cdot \text{(cylinder height)} = 20\pi r^2 \), and the volume of the cone is
\( V_2 = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 (2r) = \frac{2}{3} \pi r^3 \). Since the total volume is \( \frac{500\pi}{3} \), it follows
that \( \frac{2}{3} \pi r^3 + 20\pi r^2 = \frac{500\pi}{3} \iff r^3 + 30r^2 - 250 = 0 \). By Descartes’ Rule
of Signs, there is 1 positive zero. Since \( r \) is between 2.76 and 2.765 (see the table),
the radius should be 2.76 m (correct to two decimals).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( r^3 + 30r^2 - 250 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-219</td>
</tr>
<tr>
<td>2</td>
<td>-122</td>
</tr>
<tr>
<td>3</td>
<td>47</td>
</tr>
<tr>
<td>2.7</td>
<td>-11.62</td>
</tr>
<tr>
<td>2.76</td>
<td>-2.33</td>
</tr>
<tr>
<td>2.77</td>
<td>1.44</td>
</tr>
<tr>
<td>2.765</td>
<td>1.44</td>
</tr>
<tr>
<td>2.8</td>
<td>7.15</td>
</tr>
</tbody>
</table>

98. (a) Let \( x \) be the length, in ft, of each side of the base and let \( h \) be the height. The volume of the box is 
\( V = 2\sqrt{2} = hx^2 \), and so \( hx^2 = 2\sqrt{2} \). The length of the diagonal on the base is \( \sqrt{x^2 + x^2} = \sqrt{2x^2} \), and hence the length of the diagonal
between opposite corners is \( \sqrt{2x^2 + h^2} = x + 1 \). Squaring both sides of the equation, we have
\( 2x^2 + h^2 = x^2 + 2x + 1 \iff h^2 = -x^2 + 2x + 1 \iff h = \sqrt{-x^2 + 2x + 1} \). Therefore,
\( 2\sqrt{2} = hx^2 = (\sqrt{-x^2 + 2x + 1})x^2 \iff (-x^2 + 2x + 1) x^4 = 8 \iff x^6 - 2x^5 - x^4 + 8 = 0 \).
(b) We graph \( y = x^6 - 2x^5 - x^4 + 8 \) in the viewing rectangle \([0, 5]\) by \([-10, 10]\), and we see that there are two solutions. In the second viewing rectangle, \([1.4, 1.5]\) by \([-1, 1]\), we see the solution \( x \approx 1.45 \). The third viewing rectangle, \([2.25, 2.35]\) by \([-1, 1]\), shows the solution \( x \approx 2.31 \).

If \( x = \text{width} = \text{length} = 1.45 \text{ ft} \), then height = \( \sqrt{-x^2 + 2x + 1} = 1.34 \text{ ft} \), and if \( x = \text{width} = \text{length} = 2.31 \text{ ft} \), then height = \( \sqrt{-x^2 + 2x + 1} = 0.53 \text{ ft} \).

99. Let \( b \) be the width of the base, and let \( l \) be the length of the box. Then the length plus girth is \( l + 4b = 108 \), and the volume is \( V = lb^2 = 2200 \). Solving the first equation for \( l \) and substituting this value into the second equation yields

\[
l = 108 - 4b \Rightarrow V = (108 - 4b) b^2 = 2200 \iff 4b^3 - 108b^2 + 2200 = 0 \iff 4(b^3 - 27b^2 + 550) = 0.
\]

Now \( P(b) = b^3 - 27b^2 + 550 \) has two variations in sign, so there are 0 or 2 positive real zeros. We also observe that since \( l > 0, b < 27 \), so \( b = 27 \) is an upper bound. Thus the possible positive rational real zeros are \( 1, 2, 3, 10, 11, 22, 25 \).

\[
\begin{array}{c|cccc}
1 & 1 & -27 & 0 & 550 \\
1 & -26 & -26 & 524 \\
5 & 1 & -27 & 0 & 550 \\
5 & -110 & -550 \\
1 & -22 & -110 & 0
\end{array}
\]

\[
P(b) = (b - 5)(b^2 - 22b - 110).
\]

The other zeros are \( b = \frac{22 \pm \sqrt{484 - 4(-110)}}{2} = \frac{22 \pm \sqrt{924}}{2} = \frac{22 \pm 30.397}{2} \). The positive answer from this factor is \( b \approx 26.20 \). Thus we have two possible solutions, \( b = 5 \) or \( b \approx 26.20 \). If \( b = 5 \), then \( l = 108 - 4(5) = 88 \); if \( b \approx 26.20 \), then \( l = 108 - 4(26.20) = 3.20 \). Thus the length of the box is either 88 in. or 3.20 in.

100. (a) An odd-degree polynomial must have a real zero. The end behavior of such a polynomial requires that the graph of the polynomial heads off in opposite directions as \( x \to \infty \) and \( x \to -\infty \). Thus the graph must cross the \( x \)-axis.

(b) There are many possibilities one of which is \( P(x) = x^4 + 1 \).

(c) \( P(x) = x(x - \sqrt{2})(x + \sqrt{2}) = x^3 - 2x \).

(d) \( P(x) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3}) = x^4 - 5x^2 + 6 \). If a polynomial with integer coefficients has no real zeroes, then the polynomial must have even degree.
101. (a) Substituting $X - \frac{a}{3}$ for $x$ we have

$$x^3 + ax^2 + bx + c = \left(X - \frac{a}{3}\right)^3 + a \left(X - \frac{a}{3}\right)^2 + b \left(X - \frac{a}{3}\right) + c$$

$$= X^3 - aX^2 + \frac{a^2}{3}X + \frac{a^3}{27} + a \left(X - \frac{a}{3}\right)^2 + a \left(X - \frac{2a}{3}X + \frac{a^2}{9}\right) + bX - \frac{ab}{3} + c$$

$$= X^3 - aX^2 + \frac{a^2}{3}X + \frac{a^3}{27} + aX^2 - \frac{2a^2}{3}X + \frac{a^3}{9} + bX - \frac{ab}{3} + c$$

$$= X^3 + (-a + a)X^2 + \left(-\frac{a^2}{3} - \frac{2a^2}{3} + b\right)X + \left(\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c\right)$$

$$= X^3 + (b - a^2)X + \left(\frac{4a^3}{27} - \frac{ab}{3} + c\right)$$

(b) $x^3 + 6x^2 + 9x + 4 = 0$. Setting $a = 6, b = 9, c = 4$, we have: $X^3 + (9 - 6^2)X + (32 - 18 + 4) = X^3 - 27X + 18$.

102. (a) Using the cubic formula, $x = \sqrt[3]{\frac{-2}{2} + \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} + \sqrt[3]{\frac{-2}{2} - \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} = \sqrt[3]{-1} + \sqrt[3]{-1} = -1 = -2$.

So $(x + 2)(x^2 - 2x + 1) = (x + 2)(x - 1)^2 = 0 \Rightarrow x = -2, 1$. Using the methods from this section, we have

$$
\begin{array}{c|ccc}
1 & 1 & 0 & -3 & 2 \\
\hline
-2 & 1 & 0 & -3 & 2 \\
1 & -2 & 1 & 0 \\
\end{array}
$$

So $x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)^2(x + 2) = 0 \Leftrightarrow x = -2, 1$.

Since this factors easily, the factoring method was easier.

(b) Using the cubic formula,

$$x = \sqrt[3]{\frac{-(-54)}{2} + \sqrt{\frac{(-54)^2}{4} + \frac{(-27)^3}{27}}} + \sqrt[3]{\frac{-(-54)}{2} - \sqrt{\frac{(-54)^2}{4} + \frac{(-27)^3}{27}}} = \sqrt[3]{\frac{54}{2} + \sqrt{27^2 - 27^2}} + \sqrt[3]{\frac{54}{2} - \sqrt{27^2 - 27^2}} = \sqrt[3]{27} + \sqrt[3]{27} = 3 + 3 = 6$$

$$
\begin{array}{c|ccc}
6 & 1 & 0 & -27 & -54 \\
\hline
6 & 36 & 54 \\
1 & 6 & 9 & 0 \\
\end{array}
$$

$x^3 - 27x - 54 = (x - 6)(x^2 + 6x + 9) = (x - 6)(x + 3)^2 = 0 \Rightarrow x = -3, 6$.

Using methods from this section,

$$
\begin{array}{c|ccc}
-1 & 1 & 0 & -27 & -54 \\
-1 & 1 & 26 & 2 & 4 & 46 & -3 & 1 & 0 & -27 & -54 \\
1 & -1 & -26 & -28 & 1 & -2 & -23 & -8 & 1 & -3 & -18 & 0 \\
\end{array}
$$

So $x^3 - 27x - 54 = (x + 3)(x^2 - 3x - 18) = (x - 6)(x + 3)^2 = 0 \Leftrightarrow x = -3, 6$.

Since this factors easily, the factoring method was easier.
(c) Using the cubic formula,

\[ x = \frac{-3 - \sqrt{3} \pm \sqrt[3]{2 - \sqrt{4 - 4(1)(4)}}}{2} = \frac{-3 - \sqrt[3]{2} + \sqrt[3]{2 - \sqrt{4 - 4(1)(4)}}}{2} \]

From the graphing calculator, we see that \( P(x) = x^3 + 3x + 4 \) has one zero.

Using methods from this section, \( P(x) \) has possible rational zeros \( \pm 1, \pm 2, \pm 4 \).

\[
\begin{array}{ccc}
1 & 0 & 3 & 4 \\
1 & 1 & 4 & 4
\end{array}
\]

\[
\begin{array}{ccc}
-1 & 0 & 3 & 4 \\
-1 & 1 & 4 & 0
\end{array}
\]

\[ 1 \] is an upper bound. \( 1 - 4 \) \(
\Rightarrow x = -1 \) is a zero.

\[ P(x) = x^3 + 3x + 4 = (x + 1)(x^2 - x + 4). \]

Using the quadratic formula we have:

\[
x = \frac{1 - \sqrt{1 - 4(1)(4)}}{2} = \frac{1 - \sqrt{-15}}{2} \]

which is not a real number. Since it is not easy to see that

\[ \sqrt{-2 + \sqrt{5} + \sqrt{-2 - \sqrt{5}}} = -1 \]

we see that the factoring method was much easier.

### 3.4 Complex Numbers

1. \( 5 - 7i \): real part 5, imaginary part -7.
2. \( -6 + 4i \): real part -6, imaginary part 4.
3. \( \frac{-2 - 5i}{3} = -\frac{2}{3} - \frac{5}{3}i \): real part \(-\frac{2}{3}\), imaginary part \(-\frac{5}{3}\).
4. \( \frac{4 + 7i}{2} = 2 + \frac{7}{2}i \): real part 2, imaginary part \(\frac{7}{2}\).
5. 3: real part 3, imaginary part 0.
6. \( -\frac{1}{3} \): real part \(-\frac{1}{3}\), imaginary part 0.
7. \( -\frac{3}{5}i \): real part 0, imaginary part \(-\frac{3}{5}\).
8. \( i\sqrt{3} \): real part 0, imaginary part \(\sqrt{3}\).
9. \( \sqrt{3} + \sqrt{-4} = \sqrt{3} + 2i \): real part \(\sqrt{3}\), imaginary part 2.
10. \( 2 - \sqrt{-5} = 2 - i\sqrt{5} \): real part 2, imaginary part \(-\sqrt{5}\).
11. \( (2 - 5i) + (3 + 4i) = (2 + 3) + (-5 + 4)i = 5 - i \)
12. \( (2 + 5i) + (4 - 6i) = (2 + 4) + (5 - 6)i = 6 - i \)
13. \( (-6 + 6i) + (9 - i) = (-6 + 9) + (6 - 1)i = 3 + 5i \)
14. \( (3 - 2i) + (-5 - \frac{1}{3}i) = (3 - 5) + (-2 - \frac{1}{3}i) = -2 - \frac{7}{3}i \)
15. \( 3i + (6 - 4i) = 6 + (3i - 4i) = 6 - 7i \)
16. \( (\frac{1}{2} - \frac{1}{3}i) + (\frac{1}{2} + \frac{1}{3}i) = (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{3})i = 1 \)
17. \( (7 - \frac{1}{2}i) - (5 + \frac{2}{3}i) = (7 - 5) + (-\frac{1}{2} - \frac{2}{3})i = 2 - 2i \)
18. \( (4 + i) - (2 - 5i) = -4 + i - 2 + 5i = (-4 - 2) + (1 + 5)i = -6 + 6i \)
19. \( (-12 + 8i) - (7 + 4i) = -12 + 8i - 7 - 4i = (-12 - 7) + (8 - 4)i = -19 + 4i \)
20. \( 6i - (4 - i) = 6i - 4 + i = (6 + 1)i = -4 + 7i \)
21. \( \frac{1}{2}i - (\frac{1}{4} - \frac{1}{8}i) = -\frac{1}{2} + (\frac{1}{4} - \frac{1}{8})i = -\frac{1}{2} + \frac{1}{4}i \)
22. \( (0.1 - 1.1i) - (1.2 - 3.6i) = (0.1 - 1.2) + [(-1.1) - (-3.6)]i = -1.1 + 2.5i \)
23. \( 4(-1 + 2i) = -4 + 8i \)
24. \( 2i(\frac{1}{2} - i) = i - 2i^2 = 2 + i \)
25. \( (7 - i)(4 + 2i) = 28 + 14i - 4i - 2i^2 = (28 + 2) + (14 - 4)i = 30 + 10i \)
26. \( (5 - 3i)(1 + i) = 5 + 5i - 3i - 3i^2 = (5 + 3) + (5 - 3)i = 8 + 2i \)
27. \( (3 - 4i)(5 - 12i) = 15 - 36i - 20i + 48i^2 = (15 - 48) + (-36 - 20)i = -33 - 56i \)
28. \( (\frac{2}{3} + 12i)(\frac{1}{6} + 24i) = \frac{1}{6} + 16i + 2i + 288i^2 = (\frac{1}{6} - 288) + (16 + 2)i = -\frac{2501}{6} + 18i \)