Exponential growth and decay is a primary application behind differential equations. However, different scenarios use different types of equations. Consider the three cases below.

I. General Exponential Growth

“A positive quantity ‘y’ increases/decreases at a rate that at any time ‘t’ is proportional to the amount present.”

\[
\frac{dy}{dt} = ky
\]

Note that the model is growth if \( k > 0 \) and decay if \( k < 0 \). If \( k = 0 \), this is just stupid.

Now we shall find the general function that satisfies the differential equation above.

Example: The population of strange looking Canadians (aka Canadians) is growing at a rate proportional to its population. If the growth rate per year is 4% of the current population, how long will it take for the population to double?

Example (Half-Life): Cellium-314 decays at a rate proportional to the quantity present. Its half-life is 1612 years. How long will it take for one quarter of a given quantity to decay?
II. Restricted Growth

The above model, although applicable, can seem unrealistic in many scenarios. For instance, if we are modeling the amount of students in this class who will catch the flu, we won’t actually approach infinity. This calls for a different strategy:

“The rate of change of a quantity ‘y’ is proportional to a fixed constant A minus the amount of the quantity present.”

\[ \frac{dy}{dt} = k[A - y] \]

Now we shall find the general function that satisfies the differential equation above.

Example: Advertisers generally assume that the rate at which people hear about a product is proportional to the number of people who have not yet heard about it. Suppose that the size of a community is 15,000, that to begin when no one has heard about a product, but that after 6 days 1,500 people know about it. How long will it take for 2700 people to have heard of it?

Example: According to Newton’s Law of Cooling, a hot object cools at a rate proportional to the difference between its own temperature and that of its environment. If a roast at room temperature 68° is put into a 20° freezer, and if, after 2 hours, the temperature of the roast is 40°. What is the temperature after 5 hours? How long will it take for the temperature to fall to 21°?
III. Logistic Growth (BC for Real!)

Now we will combine the two ideas presented above to form a snazzy differential equation!

“The rate of change of a quantity is proportional to both the amount of the quantity and to the
different between a fixed constant \( A \) and its amount”

\[
\frac{dy}{dt} = ky[A - y] \quad \text{or} \quad \frac{dy}{dt} = Aky \left[ 1 - \frac{y}{A} \right]
\]

Now we shall find the general function that satisfies the differential equation above. Hmmm… I wonder if this proof will be on the upcoming test…?

*Note that \( ce^{-Akt} \rightarrow 0 \) so denominator of \( f(t) \rightarrow 1 \), and therefore \( f(t) \rightarrow A \) (carrying capacity)

Three properties that come to fruition using this model:

1. Increases slowly for a while
2. Attains a maximum when \( y = \frac{A}{2} \)
3. Decreases as approaches 0 and carrying capacity
Example: Because of limited food and space, a squirrel population cannot exceed 1000. It grows at a rate proportional both to the existing population and to the attainable additional population. If there were 100 squirrels 2 years ago, and 1 year ago the population was 400, about how many squirrels are there now?

Example: Suppose Ebola is spreading through a population of 50,000 at a rate proportional both to the number of people already infected and to the number still uninfected. If 100 people were infected yesterday and 130 are infected today:

a) Write an expression for the number of people $N(t)$ infected after $t$ days.

b) Determine how many will be infected a week from today.

c) Indicate when the virus is spreading the fastest.